Bouchet Graphs: A Generalization of Circle Graphs

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Abstract

A circle graph is an intersection graph of chords in a circle. These graphs have been introduced by Even and Itai in 1971 and were extensively studied. There are several characterizations for this class. One of them uses the concept of local complementations and was proposed by André Bouchet in 1994. In this thesis, we use the idea of this characterization to define a new class, which generalizes circle graphs, and we call it Bouchet graphs. We prove that these graphs also generalize interval graphs, and we find a characterization of the new class by 33 forbidden subgraphs, which is obtained by using a computer program. As a consequence of this characterization, we show that Bouchet graphs can be recognized in polynomial time. Finally, it is proved that there are 396 different formulations of Bouchet's characterization theorem for circle graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. The *neighborhood* of a vertex $v \in V(G)$ is the set N(v), which consists of all the adjacent vertices of v. Let F be a finite family of nonempty sets. The *intersection graph* of F is obtained by representing each set in F by a vertex, and connecting two vertices by an edge if and only if their corresponding sets intersect. *Circle graphs* are the intersection graphs of chords in a circle. *Interval graphs* are the intersection graphs of intervals in the real line. A graph H is a *forbidden subgraph* for a graph class C if no graph in C contains H as an induced subgraph. An undirected graph G is called *triangulated* if every cycle of length strictly grater than 3 possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle. A graph $G = (V(G), E(G))$ is a comparability graph if their edges can be oriented in such a way that the resulting directed graph $G' = (V(G), D(G))$ satisfies the following condition: $(u, v) \in D(G), (v, w) \in D(G) \Rightarrow (u, w) \in D(G)$.

The *local complementation* of a graph G at a vertex $v \in V(G)$, defined in [1] and denoted by $LC(G, v)$, is the operation which replaces the subgraph of G induced by $N(v)$ by its complement. Two graphs are *locally equivalent* if one of them can be obtained from the other by a sequence of local complementations.

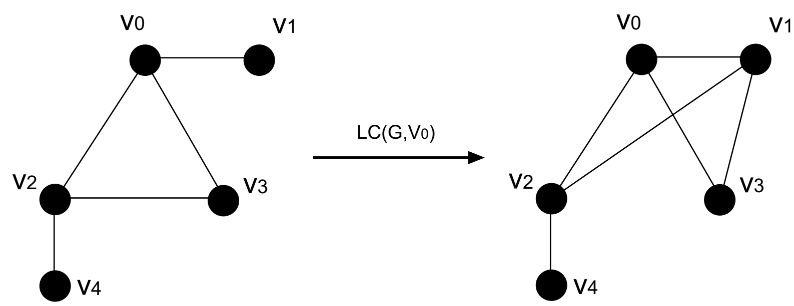

Figure 1: Local complementation.

The following properties of the local complementation can be easily proved:

Property 1 *Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. Then $LC(LC(G, v), v) = G$.*

Property 2 *Let G and H be two graphs. G is locally equivalent to H if and only if H is locally equivalent to G.*

Property 3 *Let G, H and I be three graphs. If G is locally equivalent to H, and H is locally equivalent to I, then G is locally equivalent to I.*

Let G and H be two graphs, such that G contains H as an induced subgraph. Let G' and H' be the resulting graphs of applying local complementation to G and H, respectively, at the same vertex. The following property states that G' also contains H' as an induced subgraph.

Property 4 *Let G be a graph of m vertices $\{v_1, v_2, ..., v_m\}$ and H a graph of n vertices $(m > n)$, such that H is isomorphic to the subgraph induced by $\{v_1, v_2, ..., v_n\}$ in G. Let $\{w_1, w_2, ..., w_n\}$ be the vertices of H such that the isomorphism holds after replacing v_i by w_i. Then, the subgraph induced by $\{v_1, v_2, ..., v_n\}$ in $LC(G, v_i)$ is isomorphic to $LC(H, w_i)$.*

Proof: We show that $(v_j, v_k) \in E(LC(G, v_i))$ if and only if $(w_j, w_k) \in E(LC(H, w_i))$, for $i, j, k \leq n$.

Case 1: $v_j, v_k \in N(v_i)$

Since $v_j, v_k \in N(v_i)$ and H is isomorphic to the subgraph induced by $\{v_1, v_2, ..., v_n\}$ in G, then $w_j, w_k \in N(w_i)$. Therefore,

$$(v_j, v_k) \in E(LC(G, v_i)) \Leftrightarrow (v_j, v_k) \notin E(G) \Leftrightarrow$$
$$(w_j, w_k) \notin E(H) \Leftrightarrow (w_j, w_k) \in E(LC(H, w_i))$$

Case 2: $v_j \notin N(v_i)$ or $v_k \notin N(v_i)$

Since $v_j \notin N(v_i)$ or $v_k \notin N(v_i)$ and H is isomorphic to the subgraph induced by $\{v_1, v_2, ..., v_n\}$ in G, then $w_j \notin N(w_i)$ or $w_k \notin N(w_i)$. Thus,

$$(v_j, v_k) \in E(LC(G, v_i)) \Leftrightarrow (v_j, v_k) \in E(G) \Leftrightarrow$$
$$(w_j, w_k) \in E(H) \Leftrightarrow (w_j, w_k) \in E(LC(H, w_i))$$

□

2 Bouchet graphs

There is no known characterization of circle graphs by forbidden subgraphs. Bouchet presents in [1] a characterization that may contribute in this way, using the definition of locally equivalent graphs.

Theorem 1 *A graph G is a circle graph if and only if no graph locally equivalent to G has an induced subgraph isomorphic to one of the graphs depicted in Figure 2.*

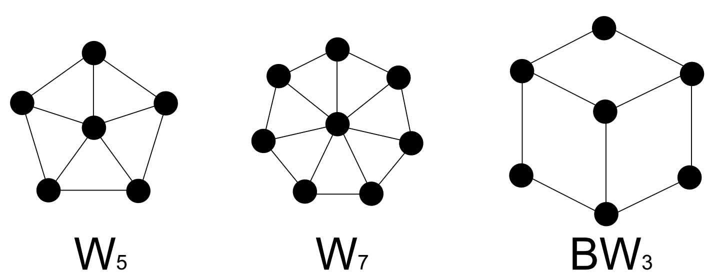

Figure 2: Graphs of Bouchet's Theorem.

We define *Bouchet graphs* as follows:

Definition 1 *A graph G is a Bouchet Graph if and only if no induced subgraph of G is locally equivalent to W_5, BW_3 or W_7 (Figure 2).*

2.1 Generalization of circle graphs

First we prove that this new class is a generalization of circle graphs:

Theorem 2 *Let G be a circle graph. Then, G is a Bouchet graph.*

Proof: Suppose that G is not a Bouchet graph. Then, it must have an induced subgraph locally equivalent to W_5, W_7 or BW_3. Let H be such a subgraph and suppose that it is locally equivalent to W_5. There exists a sequence of k vertices $v_{i_1}, v_{i_2}, ..., v_{i_k}$ such that $LC(...LC(LC(H, v_{i_1}), v_{i_2})..., v_{i_k}) = W_5$. Thus, by Property 4, the graph $G' = LC(...LC(LC(G, v_{i_1}), v_{i_2})..., v_{i_k})$ has W_5 as an induced subgraph. As G' is locally equivalent to G, by Theorem 1, we obtain that G is not a circle graph.

Note that no peculiarity of W_5 was used in this proof. It would be the same to use W_7 or BW_3 instead. □

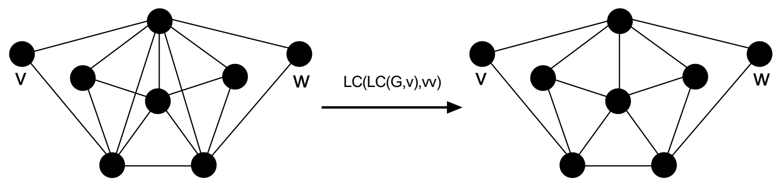

Figure 3: Non-circle Bouchet graph.

Circle graphs are a proper subset of Bouchet graphs. Figure 3 shows a graph which is not a circle graph —after applying local complementation first at vertex v and then at vertex w, the resulting graph has W_5 as an induced subgraph—, but which is an interval graph [3]. As it will be shown later, interval graphs are also generalized by Bouchet graphs; therefore, this graph is a Bouchet graph.

2.2 Characterization by forbidden subgraphs

It follows from the definition of Bouchet graphs that W_5, BW_3 and W_7 are forbidden subgraphs for this class. A simple procedure to obtain more forbidden subgraphs is to make local complementations of known forbidden graphs, using any of their vertices. Using a BFS algorithm, we closed under local complementation W_5, BW_3 and W_7, obtaining three families F_{W_5}, F_{BW_3} and F_{W_7}. The union of these families is the set of forbidden subgraphs which characterizes Bouchet graphs.

2.2.1 Family F_{W_5}

When we apply local complementation to W_5, we must consider v_0 on the one hand, and the remaining vertices on the other (taking into account the labels of the vertices of Figure 4(a)). It is clear that after applying local complementation at vertices v_1, v_2, v_3, v_4 and v_5 the resulting graphs are all isomorphic. As shown in Figure 4(b), the local complementation of W_5 at v_0 is isomorphic to W_5.

The local complementation of W_5 at v_1 yields the graph of Figure 5, which we call W_{5_1}. The local complementation of W_{5_1} at any vertex yields W_5 again.

The result of closing W_5 under local complementation is the family F_{W_5}, composed of the two graphs of Figure 6. The transition between its members by local complementation is given in Table 1 [1].

[1] If entry (i, j) of the transition table is G, it means that after applying local complementation to graph i at vertex j, the graph obtained is isomorphic to G —not necessarily with the same labelling of vertices.

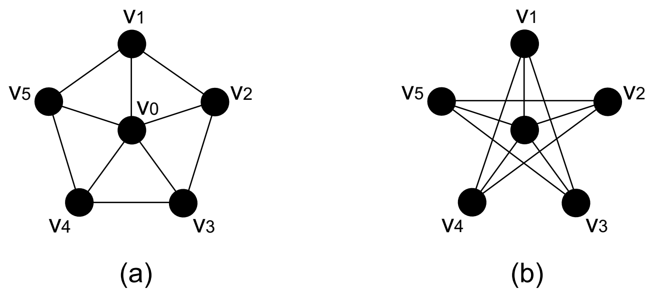

Figure 4: (a) Graph W_5; (b) $LC(W_5, v_0)$.

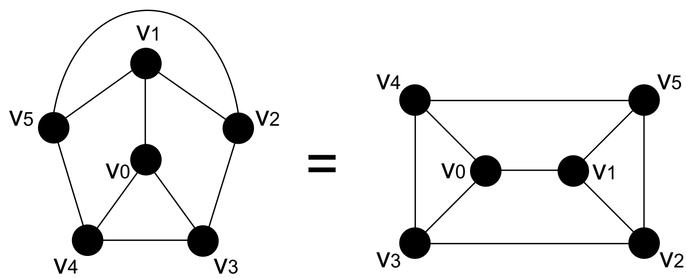

Figure 5: Local complementation of W_5 at v_1.

2.2.2 Families F_{BW_3} and F_{W_7}

In the same way, we closed under local complementation BW_3 and W_7, obtaining families F_{BW_3} (9 graphs) and F_{W_7} (22 graphs). These results are summarized in Figures 7, 8 and 9, and in Tables 2 and 3. They were obtained using a computer program.

A characterization by a finite set of forbidden subgraphs, each of them with a bounded amount of vertices, gives us a polynomial time recognition algorithm for Bouchet graphs. A graph G belongs to this class if and only if no induced subgraph of G is isomorphic to any graph in $F_{W_5} \cup F_{BW_3} \cup F_{W_7}$. As the number of vertices of these graphs is bounded by 8, this can be checked in $O(n^8)$.

2.3 Minimal set

The union of families F_{W_5}, F_{BW_3} and F_{W_7} conforms a minimal set of forbidden subgraphs that characterizes Bouchet graphs. In order to prove the minimality of the set, we must show that none of its forbidden subgraphs is a proper induced subgraph of other one. Considering the amount of vertices of the graphs of each family, it is enough to see that no graph in F_{W_5} is an induced subgraph of any graph in $F_{BW_3} \cup F_{W_7}$, and that no graph in F_{BW_3} is an induced subgraph of any graph in F_{W_7}. Moreover, by Lemmas 1 and 2, it is only necessary to prove that neither W_5 is an induced subgraph of any graph in $F_{BW_3} \cup F_{W_7}$, nor BW_3 is an induced subgraph of any graph in F_{W_7}.

Lemma 1 *Let $G \in F_{W_5}$. If G is an induced subgraph of a graph $H \in F_{BW_3} \cup F_{W_7}$, then for every graph $G' \in F_{W_5}$ there is a graph $H' \in F_{BW_3} \cup F_{W_7}$ such that G' is an induced subgraph of H'.*

Proof: Suppose $G \in F_{W_5}$ is an induced subgraph of $H \in F_{BW_3} \cup F_{W_7}$.

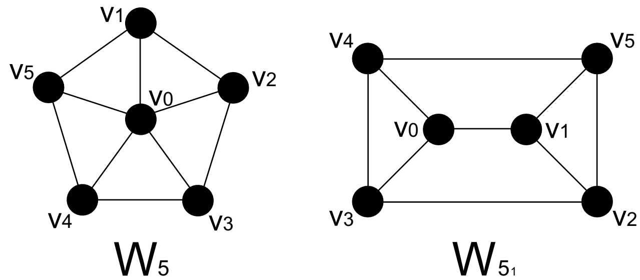

Figure 6: Graphs of family F_{W_5}.

	v_0	v_1	v_2	v_3	v_4	v_5
W_5	W_5	W_{5_1}	W_{5_1}	W_{5_1}	W_{5_1}	W_{5_1}
W_{5_1}	W_5	W_5	W_5	W_5	W_5	W_5

Table 1: Transitions of Family F_{W_5}.

Let $\{v_1, v_2, ..., v_6\}$ be the vertices of G, and $\{w_1, w_2, ..., w_6\}$ the vertices of H that induce the subgraph isomorphic to G. Suppose also that the isomorphism holds after replacing v_i by w_i $(1 \leq i \leq 6)$.

Since both G and G' belong to F_{W_5}, there exists a sequence of k vertices v_{i_1}, v_{i_2} ..., v_{i_k} of G such that: $G' = LC(...LC(LC(G, v_{i_1}), v_{i_2})..., v_{i_k})$. By Property 4, the graph $H' = LC(...LC(LC(H, w_{i_1}), w_{i_2})..., w_{i_k})$ has G' as an induced subgraph. As H' is locally equivalent to H, H' belongs to $F_{BW_3} \cup F_{W_7}$. □

The following lemma can be proved analogously:

Lemma 2 *Let $G \in F_{BW_3}$. If G is an induced subgraph of a graph $H \in F_{W_7}$, then for every graph $G' \in F_{W_5}$ there is a graph $H' \in F_{W_7}$ such that G' is an induced subgraph of H'.* □

Using a computer program we could verify that none of the graphs in $F_{BW_3} \cup F_{W_7}$ has W_5 as an induced subgraph, and none of the graphs in F_{W_7} has BW_3 as an induced subgraph. Thus, the set $F_{W_5} \cup F_{BW_3} \cup F_{W_7}$ is a minimal set of forbidden subgraphs.

2.4 Generalization of interval graphs

Interval graphs were characterized by an infinite set of forbidden subgraphs in [4]. The following theorem, due to Gilmore and Hoffman[2], provides another characterization for this class:

Theorem 3 *A graph G is an interval graph if and only if G is triangulated and $\overline{G}$ is a comparability graph.*

Using this theorem we prove that interval graphs are a subclass of Bouchet graphs.

Theorem 4 *Let G be an interval graph. Then G is a Bouchet graph.*

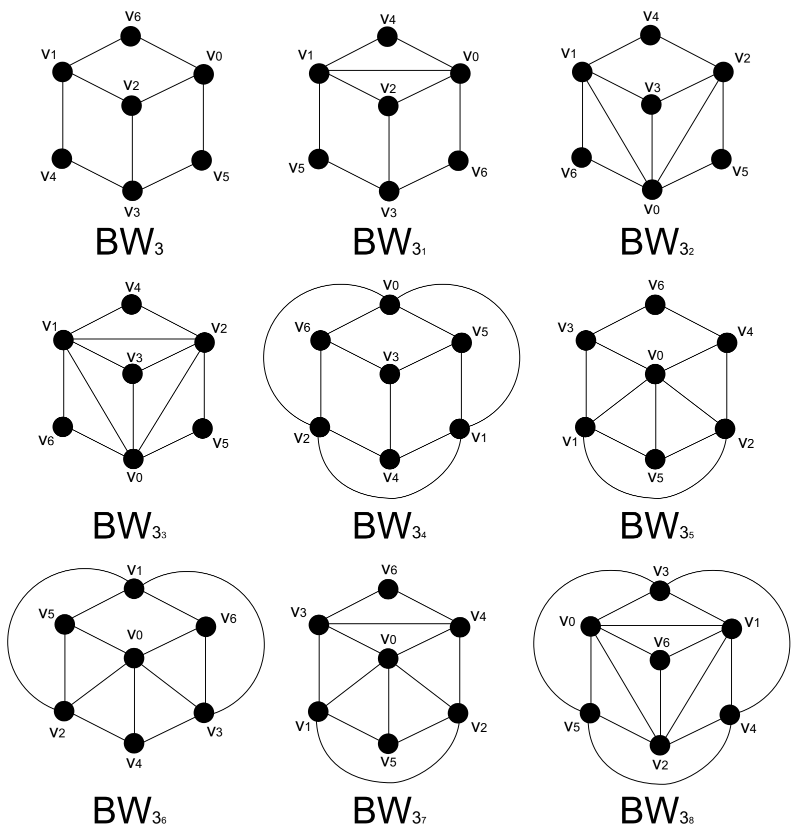

Figure 7: Family F_{BW_3}.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6
BW_3	BW_{3_5}	BW_{3_5}	BW_{3_3}	BW_{3_5}	BW_{3_1}	BW_{3_1}	BW_{3_1}
BW_{3_1}	BW_{3_5}	BW_{3_5}	BW_{3_2}	BW_{3_7}	BW_3	BW_{3_2}	BW_{3_2}
BW_{3_2}	BW_{3_7}	BW_{3_7}	BW_{3_7}	BW_{3_1}	BW_{3_3}	BW_{3_1}	BW_{3_1}
BW_{3_3}	BW_{3_5}	BW_{3_5}	BW_{3_5}	BW_3	BW_{3_2}	BW_{3_2}	BW_{3_2}
BW_{3_4}	BW_{3_4}	BW_{3_4}	BW_{3_4}	BW_{3_8}	BW_{3_6}	BW_{3_6}	BW_{3_6}
BW_{3_5}	BW_{3_3}	BW_{3_1}	BW_{3_1}	BW_{3_6}	BW_{3_6}	BW_3	BW_{3_7}
BW_{3_6}	BW_{3_8}	BW_{3_8}	BW_{3_7}	BW_{3_7}	BW_{3_4}	BW_{3_5}	BW_{3_5}
BW_{3_7}	BW_{3_2}	BW_{3_2}	BW_{3_2}	BW_{3_6}	BW_{3_6}	BW_{3_1}	BW_{3_5}
BW_{3_8}	BW_{3_6}	BW_{3_6}	BW_{3_6}	BW_{3_6}	BW_{3_6}	BW_{3_6}	BW_{3_4}

Table 2: Transitions of family F_{BW_3}.

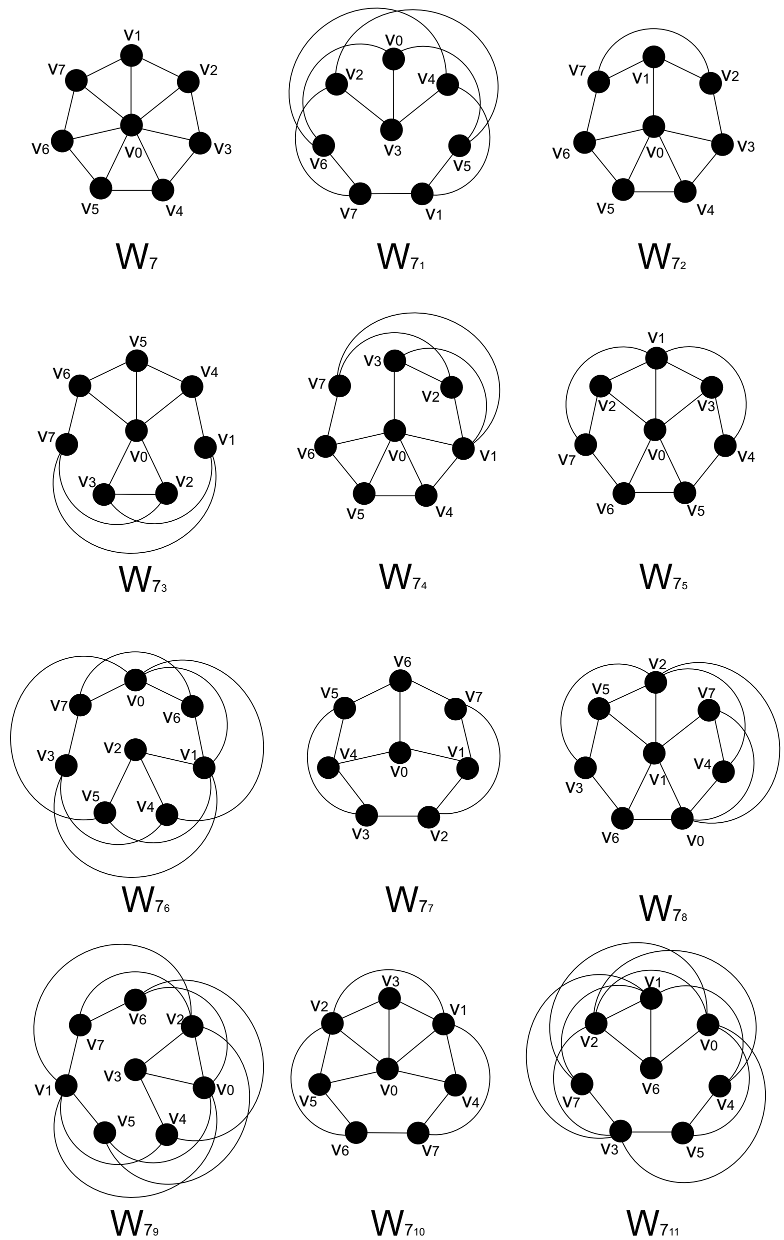

Figure 8: First 16 graphs of family F_{W_7}.

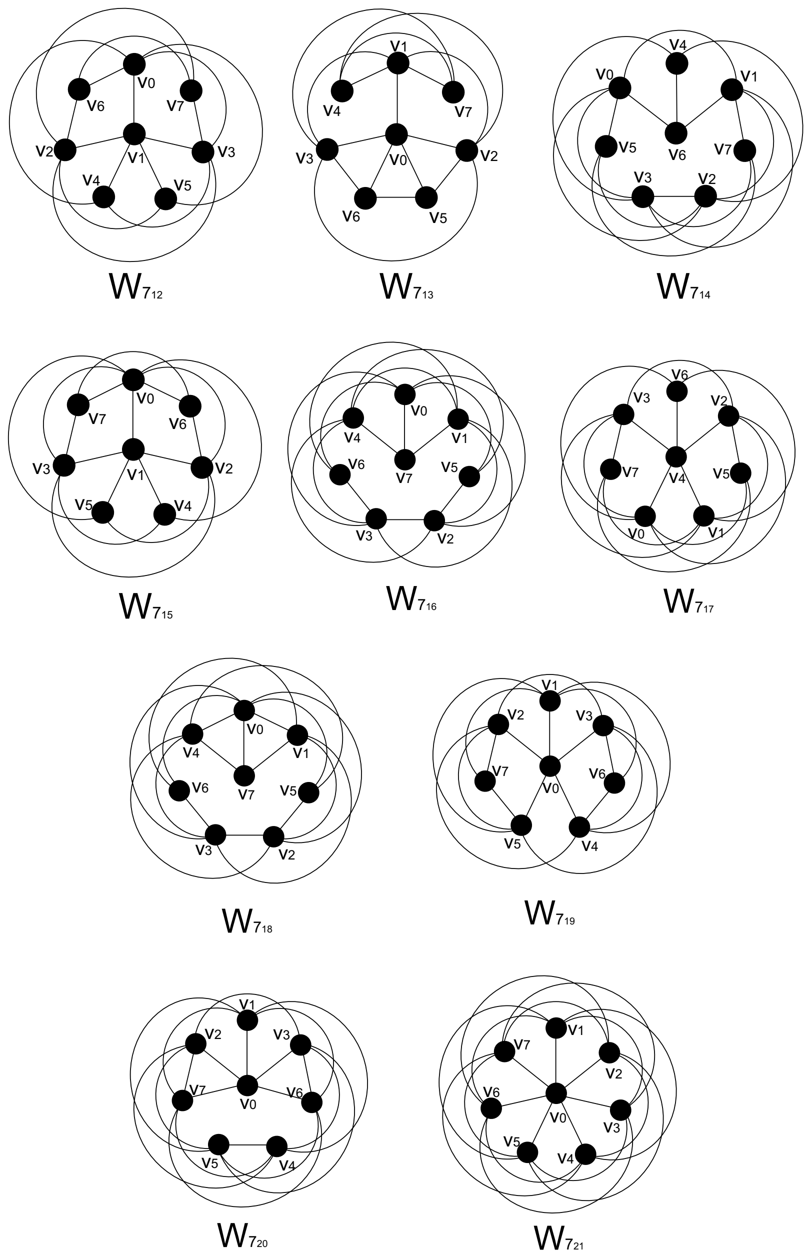

Figure 9: Last 4 graphs of family F_{W_7}.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
W_7	$W_{7_{21}}$	W_{7_2}	W_{7_2}	W_{7_2}	W_{7_2}	W_{7_2}	W_{7_2}	W_{7_2}
W_{7_1}	W_{7_9}	W_{7_9}	W_{7_9}	W_{7_9}	W_{7_9}	W_{7_9}	W_{7_9}	W_{7_9}
W_{7_2}	$W_{7_{15}}$	W_7	W_{7_4}	W_{7_5}	W_{7_7}	W_{7_7}	W_{7_5}	W_{7_4}
W_{7_3}	$W_{7_{17}}$	$W_{7_{12}}$	W_{7_4}	W_{7_4}	W_{7_6}	W_{7_7}	W_{7_6}	$W_{7_{12}}$
W_{7_4}	$W_{7_{12}}$	$W_{7_{12}}$	W_{7_2}	W_{7_3}	W_{7_3}	W_{7_2}	W_{7_8}	W_{7_8}
W_{7_5}	$W_{7_{19}}$	$W_{7_{13}}$	W_{7_2}	W_{7_2}	W_{7_8}	$W_{7_{10}}$	$W_{7_{10}}$	W_{7_8}
W_{7_6}	$W_{7_{18}}$	$W_{7_{18}}$	W_{7_8}	$W_{7_{15}}$	$W_{7_{15}}$	W_{7_3}	W_{7_3}	W_{7_8}
W_{7_7}	$W_{7_{10}}$	W_{7_2}	W_{7_3}	W_{7_3}	W_{7_2}	W_{7_2}	$W_{7_{10}}$	W_{7_2}
W_{7_8}	$W_{7_{10}}$	$W_{7_{17}}$	$W_{7_{16}}$	$W_{7_{13}}$	W_{7_5}	W_{7_4}	$W_{7_{12}}$	W_{7_6}
W_{7_9}	$W_{7_{15}}$	$W_{7_{15}}$	$W_{7_{10}}$	$W_{7_{11}}$	$W_{7_{18}}$	W_{7_1}	$W_{7_{18}}$	$W_{7_{11}}$
$W_{7_{10}}$	W_{7_9}	W_{7_8}	W_{7_8}	W_{7_7}	W_{7_5}	W_{7_5}	$W_{7_{12}}$	$W_{7_{12}}$
$W_{7_{11}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{12}}$	$W_{7_{12}}$	$W_{7_{16}}$	$W_{7_{16}}$	W_{7_9}	W_{7_9}
$W_{7_{12}}$	$W_{7_{18}}$	W_{7_4}	$W_{7_{19}}$	$W_{7_{11}}$	W_{7_3}	W_{7_8}	$W_{7_{15}}$	$W_{7_{10}}$
$W_{7_{13}}$	W_{7_5}	W_{7_5}	$W_{7_{14}}$	$W_{7_{14}}$	W_{7_8}	W_{7_8}	W_{7_8}	W_{7_8}
$W_{7_{14}}$	$W_{7_{13}}$	$W_{7_{13}}$	$W_{7_{13}}$	$W_{7_{13}}$	$W_{7_{17}}$	$W_{7_{17}}$	$W_{7_{17}}$	$W_{7_{17}}$
$W_{7_{15}}$	$W_{7_{18}}$	W_{7_2}	W_{7_9}	W_{7_9}	W_{7_6}	W_{7_6}	$W_{7_{12}}$	$W_{7_{12}}$
$W_{7_{16}}$	$W_{7_{21}}$	$W_{7_{17}}$	W_{7_8}	W_{7_8}	$W_{7_{17}}$	$W_{7_{11}}$	$W_{7_{11}}$	$W_{7_{18}}$
$W_{7_{17}}$	$W_{7_{16}}$	$W_{7_{16}}$	W_{7_8}	W_{7_8}	W_{7_3}	$W_{7_{19}}$	$W_{7_{14}}$	$W_{7_{19}}$
$W_{7_{18}}$	$W_{7_{15}}$	$W_{7_{12}}$	W_{7_6}	W_{7_6}	$W_{7_{12}}$	W_{7_9}	W_{7_9}	$W_{7_{16}}$
$W_{7_{19}}$	W_{7_5}	$W_{7_{20}}$	$W_{7_{11}}$	$W_{7_{11}}$	$W_{7_{12}}$	$W_{7_{12}}$	$W_{7_{17}}$	$W_{7_{17}}$
$W_{7_{20}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$	$W_{7_{19}}$
$W_{7_{21}}$	W_7	$W_{7_{16}}$	$W_{7_{16}}$	$W_{7_{16}}$	$W_{7_{16}}$	$W_{7_{16}}$	$W_{7_{16}}$	$W_{7_{16}}$

Table 3: Transitions of family F_{W_7}.

Proof: As we will see, none of the 33 forbidden subgraphs is an interval graph. Then, if a graph G is an interval graph, it cannot contain one of the 33 forbidden subgraphs as an induced subgraph. Therefore, G is necessarily a Bouchet graph.

Graph BW_{3_3} is the only triangulated graph in the set of forbidden subgraphs. Thus, by Theorem 3, none of the 32 remaining graphs is an interval graph.

We still have to see that BW_{3_3} is not an interval graph. In order to do it, we show that $\overline{BW_{3_3}}$ is not a comparability graph, and again by Theorem 3, BW_{3_3} is not an interval graph.

It is not possible to find an orientation of the edges of $\overline{BW_{3_3}}$ such that it satisfies the comparability property. Figure 10(a) shows $\overline{BW_{3_3}}$. It is easy to see that there is no difference between any of both possible orientations of edge (v_4, v_6). Considering orientation $v_6 \to v_4$ (Figure 10(b)), (v_2, v_6) and (v_0, v_4) must be orientated $v_6 \to v_2$ and $v_0 \to v_4$ respectively (Figure 10(c)). Then, the orientation of (v_4, v_5) and (v_5, v_6) must necessarily be $v_5 \to v_4$ and $v_6 \to v_5$ (Figure 10(d)). Now, it is not possible to orient (v_1, v_5) maintaining the comparability property. $\square$

Clearly, interval graphs are a proper subset of Bouchet graphs. For instance, C_4 (a simple cycle formed by 4 vertices) is a Bouchet graph, but not an interval graph.

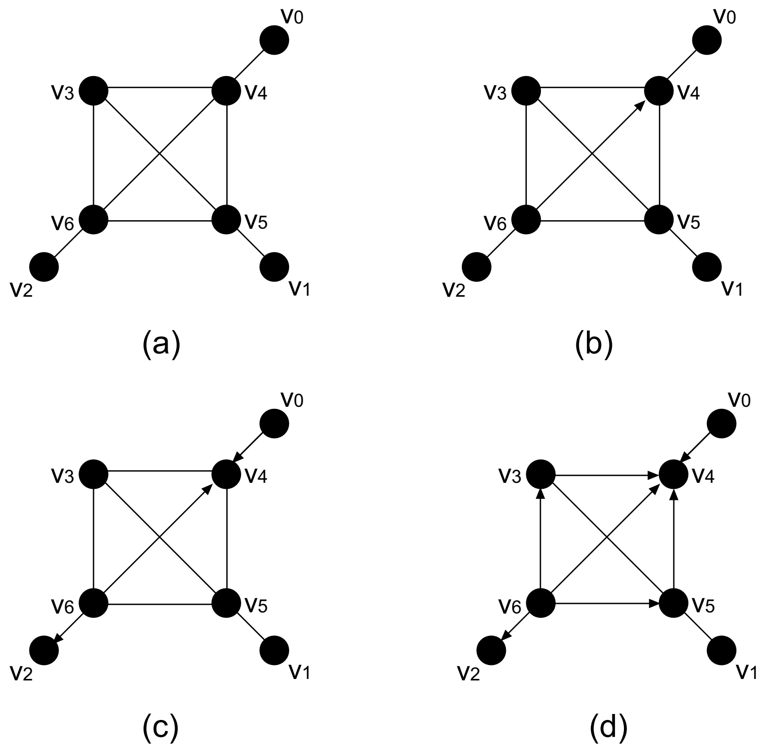

Figure 10: $\overline{BW_{3_3}}$ is not a comparability graph.

3 Generalization of Bouchet's theorem

Using families F_{W_5}, F_{BW_3} and F_{W_7}, we can prove that there are 396 equivalent formulations of Theorem 1. First, we need the following lemma:

Lemma 3 *Let G be any graph and $G_{W_5} \in F_{W_5}$. Then, G is locally equivalent to a graph H that has W_5 as an induced subgraph if and only if G is locally equivalent to a graph H' that has G_{W_5} as an induced subgraph.*

Proof: As $G_{W_5} \in F_{W_5}$, there exists a sequence of k vertices $v_{i_1}, v_{i_2}, ..., v_{i_k}$ such that $LC(...LC(LC(W_5, v_{i_1}), v_{i_2})..., v_{i_k}) = G_{W_5}$.

$\Rightarrow$) Suppose that G is locally equivalent to a graph H that has W_5 as an induced subgraph. As W_5 and G_{W_5} are locally equivalent, by Property 4, the graph $H' = LC(...LC(LC(H, v_{i_1}), v_{i_2})..., v_{i_k})$ has G_{W_5} as an induced subgraph. Then, by Property 3, G is locally equivalent to H'.

$\Leftarrow$) Now suppose that G is locally equivalent to a graph H' that has G_{W_5} as an induced subgraph. As W_5 and G_{W_5} are locally equivalent, by Property 4, the graph $H = LC(LC(...LC(H', v_{i_k})..., v_{i_2}), v_{i_1})$ has W_5 as an induced subgraph. Then, by Property 3, G is locally equivalent to H. □

The following lemmas can be proved analogously:

Lemma 4 *Let G be any graph and $G_{BW_3} \in F_{BW_3}$. Then, G is locally equivalent to a graph H that has BW_3 as an induced subgraph if and only if G is locally equivalent to a graph H' that has G_{BW_3}*

as an induced subgraph.

Lemma 5 *Let G be any graph and $G_{W_7} \in F_{W_7}$. Then, G is locally equivalent to a graph H that has W_7 as an induced subgraph if and only if G is locally equivalent to a graph H' that has G_{W_7} as an induced subgraph.*

Finally, the generalization of the characterization for circle graphs is enunciated in the following theorem:

Theorem 5 *There are 396 possible formulations of Bouchet's theorem for circle graphs, replacing W_5 by any graph in F_{W_5} (2 graphs), BW_3 by any graph in F_{BW_3} (9 graphs), and W_7 by any graph in F_{W_7} (22 graphs).*

Proof: It is a consequence of Lemmas 3, 4, 5 and that $2 \times 9 \times 22 = 396$. □

4 Summary

A new class of graphs has been defined using the ideas introduced by André Bouchet in his theorem of characterization of circle graphs. A characterization by a minimal set of forbidden subgraphs for this new class was given here. It is important to remark the use of a computer program in this proof. It was also shown that this class generalizes circle and interval graphs. It remains as further work to study what happens in Bouchet graphs with those problems that in general are NP-Complete, but have polynomial time solutions for circle and interval graphs. Last, but not least, it was proved in this thesis that there are several equivalent formulations of Bouchet's original theorem for circle graphs.

References

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