Polynomial time computation of the reliability of source-terminal complete networks with path length restrictions

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Abstract:

In this paper we study the reliability of a network with two terminal nodes *s* and *t*, where its links are subject to random, independent failures (nodes are always operational), and the network is operating if the surviving edges allow the terminal nodes to be connected by a path of length bounded by a giving parameter *D* (corresponding to a constraint on communication time between those two nodes).

In particular, we express the reliability of complete graph topologies (i.e. every pair of nodes of the network are adjacent), whose links belong to four different reliability classes, as a recursive formula of smaller networks and parameter *D*. The computational complexity of a direct implementation of a network with n nodes, is of order n^D, which give us polynomial complexity for calculating the reliability for fixed values of *D*. This is an improvement over previous results since in general determining the reliability for the networks with two terminal nodes and fixed parameter D, was shown to be a NP-Hard problem [2].

In addition, this recursive formula can be applied for computing the reliability polynomial (i.e. the reliability expressed as a polynomial function of a unique link reliability) of complete networks.

Key words: network reliability, reliability polynomial, diameter, path length, delay constraints, graph theory.

Resumen:

En este trabajo se estudia la confiabilidad de una red con dos nodos terminales *s* y *t*, tal que sus líneas están sujetas a fallas aleatorias e independientes, y la red funciona si las líneas en funcionamiento permiten la conexión de los nodos terminales a través de un camino de largo acotado por un parámetro *D* (que permite modelar por ejemplo una restricción en el tiempo de comunicación entre estos nodos).

En particular, se da la confiabilidad de redes de topología completa (es decir, donde todo par de nodos es adyacente) y cuyas líneas pertenecen a cuatro clases de confiabilidad diferente, como una fórmula recursiva en términos de un número de nodos más pequeños y parámetro *D* menor. La complejidad computacional de una implementación directa de la fórmula en una red con n nodos, es de orden n^D, resultando en complejidad polinómica para valores fijos de x Esto es una mejora sobre los resultados pre-existentes, dado que en general determinar la confiabilidad de redes con dos nodos terminales y parámetro *D* fijo, es un problema NP-difícil [2].

Además, esta fórmula recursiva puede ser utilizada para calcular el polinomio de confiabilidad (es decir, la confiabilidad como función de una única confiabilidad de líneas) de grafos completos.

Key words: confiabilidad de redes, polinomio de confiabilidad, diámetro, largo de caminos, restricciones de demora, teoría de grafos.

1. Introduction

Consider an undirected graph $G = (V, E)$ consisting of a set of nodes V and a set of connecting edges E, with distinguished set of terminal nodes K. If we suppose that nodes do not fail, and that each edge e is assigned an independent probability of operation r_e (called edge reliability), this graph can model a communication network where the operational states are those where any pair of terminal nodes are connected by a path composed of operational links. This is a random event, which has probability $R_K(G)$. The problem of numerically evaluating $R_K(G)$, or its complement, $Q_K(G) = 1-R_K(G)$, is usually called the K-terminal reliability problem, and the exact evaluation of the reliability is an NP-hard problem [6]. When all the edges have the same reliability r, evaluating $R_K(G)$ as a polynomial on r, usually called the reliability polynomial problem, also belong to the NP-hard computational class. For further reading about this classical model see [3,7].

There are many situations where it is not enough to know that the terminal nodes K can be connected, but is also required that the length of the connecting paths (measured by the number of edges) between terminal nodes, is smaller or equal to a given upper bound D. This is the case for example when at each node (or at each edge) there is a transmission delay T, and the total communication time between two terminals must be less than D times this delay. If the operational paths which connect the terminals nodes have at most length D, we can guarantee a small delay, one of the most important QoS parameters relevant for many real-time network services such as voice over IP, videoconference, and multicast applications. This new reliability measure of a network G with terminal set K and parameter D, $R_K(G,D)$, is called diameter-constrained K-terminal reliability [1,5]. If the edges of the graph have the same reliability r, we use the alternative notation $R_K(G,r,D)$ for its reliability. The specific cases where $K=V$ and $K=\{s,t\}$ are called the diameter-constrained all-terminal and the diameter-constrained 2-terminal reliability, respectively.

As in general computing the diameter-constrained reliability for networks with two terminal nodes s and t, even when a fixed parameter D is under consideration, was shown to be NP-Hard [2], an interesting problem is find classes of graphs for which polynomial time algorithms exist. In this paper, we study the diameter-constrained 2-terminal reliability for complete networks.

In the next section, we determine the diameter-constrained 2-terminal reliability polynomial (i.e. a polynomial as a function a unique edge reliability r) of complete networks, for specific values of the parameter D. Section 3 introduces a more general class of probabilistic graphs, having also a complete topology but whose edge reliabilities can be partitioned into four possible values. For these graphs, we can define a reliability multinomial in terms of the reliabilities of the different classes of edges, and a recursive formula for this reliability is then obtained. Finally, in Section 4, we present conclusions and directions for future work.

2. Source-terminal networks with complete graph topology and identical edge reliabilities

An undirected graph is usually denoted by $G=(V,E)$, where $V=\{1,...,n\}$is the node-set and $E=\{e_1,...,e_m\}$ is the edge-set. We also use the notation e_{uv} to denote an edge with end-points u and v.

A graph on n nodes, is a complete graph (denoted by K_n [4]), if each pair of nodes are adjacent. We will study complete graphs with $n+2$ nodes, whose edges have identical reliabilities r, and two distinguished nodes, called source and terminal, which must be connected by a path of length at most D (see Figure 1).

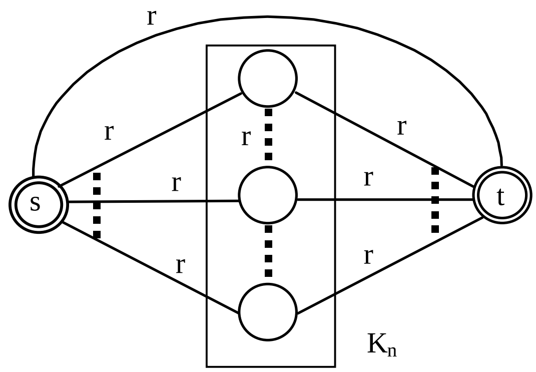

Figure 1: network with K_{n+2} topology, equireliable links, and arbitrary source and terminal nodes.

We are interested in computing the network reliability polynomial $R_{st}(K_{n+2},r,D)$, for a given value of D. If we define $P_{st}(D)$ as the set of paths between s and t, of length at most D (with length measured as the number of edges of the path), and we denote by $OP(p)$ the event in which all the edges in a single path p are operational, the network reliability polynomial can be expressed in function of these events (parametric on r):

$$\mathrm{R}_{st}(K_{n+2},r,D) = \Pr(\exists \text{ operational path of lenght} \leq \text{D from s to t}) = \Pr\left(\bigcup_{p\in Pst(D)} OP(p)\right).$$

In order to illustrate this formula, we will take the K_4 network shown in Figure 2; we will suppose that we want to establish a connection of maximum path length 2 between s and T (i.e, D=2).

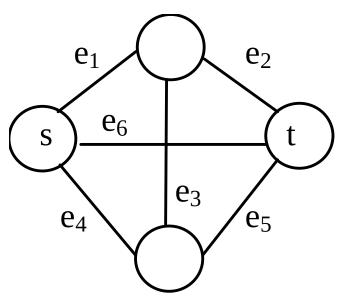

Figure 2: K_4 network

In this example, there are only three paths of length at most 2 between s and t, which will be included in the set $P_{st}(2)$ ={($e6$), (e_1,e_2) , (e_4,e_5)}. The paths (e_1,e_3,e_5) and (e_4,e_3,e_2), which connect s and t, are not considered, because they exceed the length bound. For this network, as the paths are disjoints and the events independent, we can easily compute the reliability polynomial:

$$\mathrm{Rel}_{st}(K_4,r,2) = \Pr\left(OP((e_1,e_2))\bigcup OP((e_4,e_5))\bigcup OP((e_6))\right) = r + (1-r)\left[r^2 + (1-r^2)r^2\right] =$$
$$= r + r^2 + r^2 - r^4 - r^3 - r^3 + r^5 = r + 2r^2 - 2r^3 - r^4 + r^5.$$

We will try to extend this procedure for networks K_{n+2} for arbitrary n.

First, we will look at the case where D=1. In this case, there is only one path of length 1 between s and t, namely the path including only link e_{st}. Then $\mathrm{R}_{st}(K_{n+2}, r, 1) = \Pr(OP((e_{st}))) = r$.

When D=2, the set of paths of interest is $P_{st}(2) = \{(e_{st})\} \bigcup \{(e_{sv}, e_{vt}), v \in V - \{s,t\}\}$. Like in the previous example, the paths are disjoint, and the events are independent, leading to the following expression for the reliability polynomial [1]:

$$\mathrm{R}_{st}(K_{n+2}, r, 2) = r + (1-r)\sum_{k=0}^{n-1}(1-r^2)^k r^2 = 1-(1-r)(1-r^2)^n.$$

Unfortunately, when we consider D>2, the events corresponding to the paths are neither independent from each other, nor disjoint, so that it is not possible to directly apply a similar formula for direct computation of the reliability polynomial. In the next section, we also study the diameter-constrained reliability for the class of complete networks, but here allowing up to four different edges reliability values, and, as a result, we obtain a formulation for the reliability in terms of these values, which can also be applied to compute the reliability polynomial when all links are equireliable.

3. Source-terminal networks with different classes of edge reliabilities

In this section we define an auxiliary class of complete graphs, whose edges can be partitioned into four elementary reliabilities values. We will denote these graphs by $G_A(n, r_{st}, r_s, r_t, r)$, where n is the number of intermediate nodes between the terminals s and t, r_{st} is the reliability of the edge connecting s and t, r_s is the reliability of the edges connecting s to the other n intermediate nodes, r_t is the reliability of the edges connecting T to the n intermediate nodes, and r is the reliability of the edges whose end-points are intermediate nodes. Figure 3 shows graphically a $G_A(n, r_{st}, r_s, r_t, r)$ network.

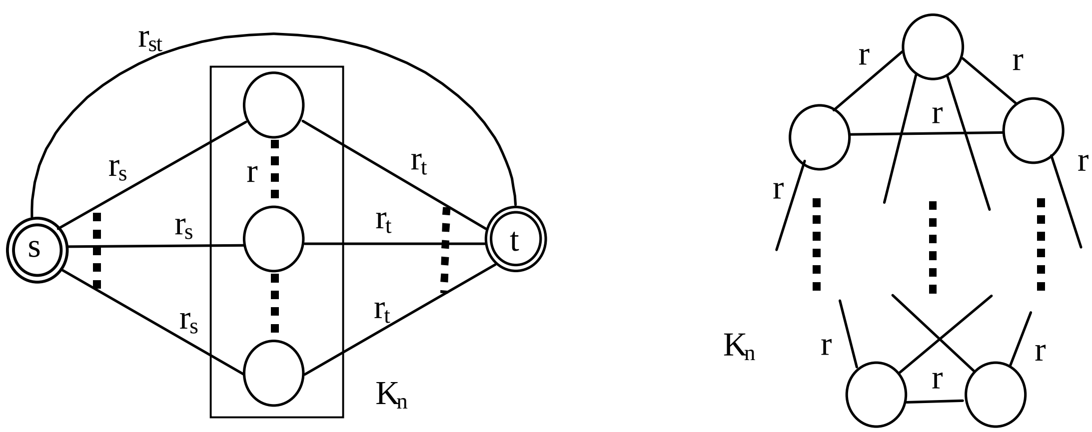

Figure 3: $G_A(n, r_{st}, r_s, r_t, r)$ network

The reliability for this network, $\mathrm{R}_{st}(G_A(n, r_{st}, r_s, r_t, r), D)$, is a multinomial in r_{st}, r_s, r_t, r. We have that the reliability for the network with identical links reliabilities is a particular case of this class of networks, i.e, that $\mathrm{R}_{st}(K_{n+2}, r, D) = \mathrm{R}_{st}(G_A(n, r, r, r, r), D)$.

We will express the reliability of network G_A by taking into account the independence between the edge (s,t) and the other paths, and by conditioning on the number of operational edges between s and the intermediate nodes (exploiting the symmetry between those nodes).

If D=1, the only feasible path is (e_{st}); then trivially $\mathrm{R}_{st}\left(G_A\left(n, r_{st}, r_s, r_t, r\right), 1\right) = r_{st}$. We can now look at the case where D>1. As a first step, we use the fact that (e_{st}) is a feasible path; and that it is independent from all other paths from s to t. Then we can write

$$\mathrm{R}_{st}\left(G_A\left(n, r_{st}, r_s, r_t, r\right), D\right) = r_{st} + \left(1 - r_{st}\right)\mathrm{R}_{st}\left(G'_A\left(n, r_s, r_t, r\right), D\right)$$

where $G'_A(n, r_s, r_t, r)$ is the probabilistic network $G_A\left(n, r_{st}, r_s, r_t, r\right)$ minus the edge e_{st} (see Figure 4).

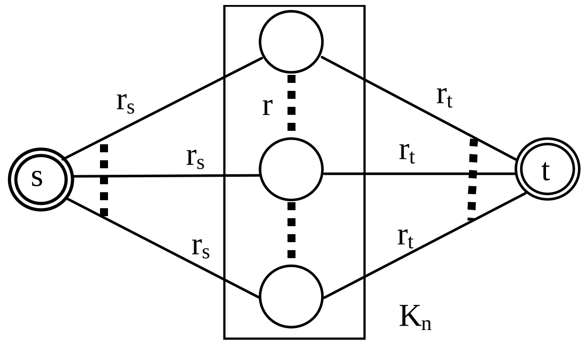

Figure 4: $G'_A(n, r_s, r_t, r)$ network

As this network is completely symmetric with respect to the edges between s and the intermediate nodes (all nodes, except s and t), we will define a partition of the probability state space for the network based on the number of those edges that fail or are operational. We define A_k to be the event where K edges from s to intermediate nodes fail, and the remaining n-k are operating; its probability is

$$P\left(A_k\right) = \binom{n}{k}\left(1 - r_s\right)^k r_s^{\,n-k}. \qquad (1)$$

The set $\left\{A_k : 0 \le k \le n\right\}$ is a partition of the probability space, as the events are pairwise disjoint, and their union has probability one. Applying the total probability theorem, we then have:

$$\mathrm{R}_{st}\left(G'_A\left(n, r_s, r_t, r\right), D\right) = \Pr\left(\exists \text{ operational path of lenght} \le \mathrm{D} \text{ from s to t in } G'_A\right) =$$
$$\sum_{k=o}^{n} \Pr\left(\exists \text{ operational path of lenght} \le D \text{ from } s \text{ to } t \text{ in } G'_A \middle| A_k\right)\Pr\left(A_k\right) \qquad (2)$$

In this last sum, the term where k=n is null, because if all links between s and the intermediate nodes fail, there is not any operational path between s and t.

We must now find and expression for the general term with K <n. The leftmost network shown in Figure 5 corresponds to this event, where we have K edges between s and intermediate nodes fail (which can be removed from the network), and n-k operational ones, which are presented with a bolder trace. Finding an operational path of length ≤ D in this network corresponds to a path of length ≤ D-1 in the network shown at the center of Figure 5, which is obtained by identifying s and the intermediate nodes to which it is unconditionally connected, into one node. In addition, as edges fail independently, the operational probability of a bank of parallel edges is the complement (to one) of the product of their failure probabilities. Thus the set of parallel edges between s and T shown at the center of Figure 5, can be replaced by a single edge with reliability 1-(1-r_t)$^{n-k}$.

Similarly, the set of parallel edges between s and an intermediate node of K_k, can be replaced by a single edge with reliability 1-(1-r)$^{n-k}$.

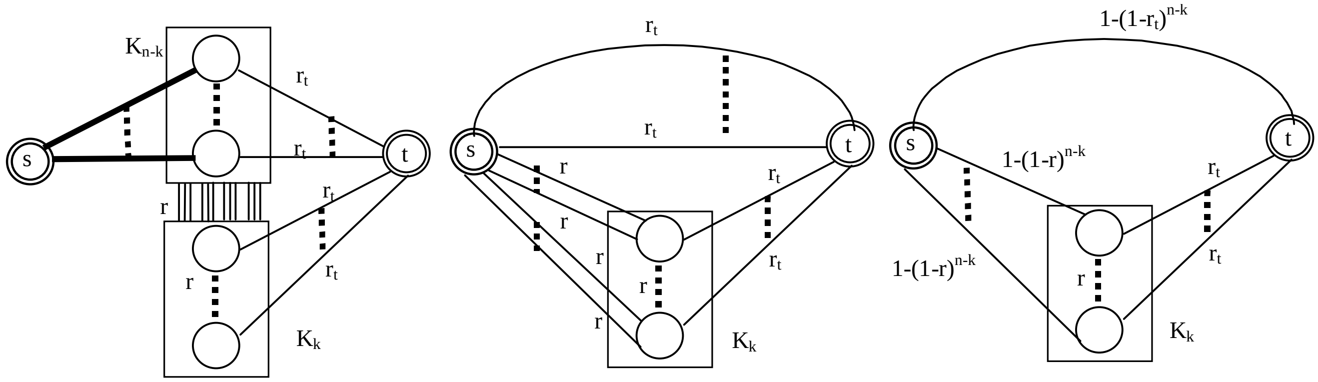

Figure 5: $G'_A(n, r_s, r_t, r)$ when K edges between s and intermediate nodes work and the rest fail.

The network resulting from this last operation is shown at the right of Figure 5, and corresponds to a $G_A\left(k, 1-(1-r_t)^{n-k}, 1-(1-r)^{n-k}, r_t, r\right)$ topology, where the operational paths must have length at most D-1.

From (1), (2) and the latest fact, we can express the reliability of the original network G_A by the following formula:

$$\mathrm{R}_{st}\left(G_A(n, r_{st}, r_s, r_t, r), D\right) = r_{st} + (1-r_{st})\sum_{k=0}^{n-1}\binom{n}{k} r_s^{\,n-k}(1-r_s)^k \mathrm{R}_{st}\left(G_A\left(k, 1-(1-r_t)^{n-k}, 1-(1-r)^{n-k}, r_t, r\right), D-1\right).$$

The recursive application of this formula gives a multinomial on r_{st}, r_s, r_t, r; in particular, if we take all these values equal to r, we have the reliability polynomial of the complete graph with n+2 nodes. A direct application of the formula has total computational complexity of order n^D, so that if D is fixed, the method will have polynomial complexity. In this sense, this is an important result since in [2] it was shown that evaluating $R_{st}(G,D)$ for any arbitrary probabilistic graph G and for fixed parameter $D > 2$, is NP-Hard.

Also, it can be observed that the direct application of the recursive formula leads to evaluating many times the multinomial $\mathrm{R}_{st}\left(G_A(k, r_{st}, r_s, r_t, r), d\right)$ for values of $k<n$ and $d<D$, but with different values to be substituted for the parameters r_{st}, r_s, r_t, and r. An alternative implementation would compute the generic multinomial for d=2 and all values of $k<n$ (using the results of Section 2). These results would then be employed for computing the case where d=3 for all values of $k<n$, substituting generic parameters for the recursive call ones; this would be repeated for d=4, etc. This way, only nD (at most, n^2) different generic multinomials would be generated. Nevertheless, the parameter substitution in itself is a costly task, and it seems that the total number of terms for the last multinomial would be again of order n^D .Thus more research is needed to simplify the multinomial generated at the intermediate steps of the recursion, and in this way reducing the total computational complexity.

4. Conclusions

We have presented a recursive formula for determining the diameter-constrained 2-terminal reliability for complete probabilistic graphs whose edge reliabilities can be partitioned into four possible values, depending if they are incident at the terminal nodes s or t. Evaluation of the reliability using this formula yields a polynomial computational complexity when fixed values of the parameter D are under consideration, and thus representing a computational improvement since it was shown that in general computing the reliability when $|K|=2$ and fixed $D > 2$ is an NP-Hard problem.

In addition, as a specific case, this recursive formula can then be applied for computing the reliability polynomial of complete networks with n nodes, and, in this way, obtaining an efficient upper bound of the reliability for an arbitrary topology on the same number of nodes, and when all its links are equally reliable (or even when we have different reliabilities for each of the four classes mentioned). This can be also used to assess the precision of approximate reliability computation (for example, by Monte Carlo methods), as well as assess the quality of heuristic optimization procedures for finding highly reliable topologies with other constraints.

One open problem is to find, if possible, a closed form analytical formula for the reliability of these complete topologies. If the latest task cannot be accomplished, it is important to develop a more efficient implementation than direct application of the recursive formula, especially for high values of D.

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