Representing SFP Domains as Information Systems

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Abstract

In this paper we exhibit a good representation for SFP domains, with the intention of keeping as close as possible the spirit of Scott's original construction of information systems and, in this form, to provide a more concrete and logical account of SFP domains. In this sense, we introduce the notion of Plotkin Information System, which is a simples generalization of Scott Information System [Sco82b], and we show its equivalence (categorical) with SFP domains. We provide constructions such as the Plotkin domain and the space of functions to show that this representation is adequate to give denotational semantics to programming languages with nondeterministic features. We use a cpo of Plotkin information systems to solve, using the technique of least fixed-point, the recursively defined SFP domain equations.
1 Introduction

The domain theory was developed by D. Scott and C. Strachey with the goal of studying a class of appropriate spaces - Scott domains - in order to resolve recursive equations, such as $D \cong D \rightarrow D$, and in this form to obtain semantic functions to give denotational semantics of programming languages [Sco82b] and [Plo83].

Scott herself introduced the notion of information system initially with the intention to make domain theory accessible to a wider audience [Sco82b]. In this representation of Scott domains the idea of information is made explicit, in the sense that each element is seen as a collection of informations that it "satisfy". Even though both notions are equivalent, categorically speaking, information systems allow to capture the logical aspects of domain, in the sense that properties of domains can be derived from assumptions about the entailment between propositions expressing properties of computations.

The notion of a powerdomain was created in order to extend Scott's framework to the simplest nondeterministic features of programming languages. The interest of the latter is clear; that of the former arises when we try to reduce parallelism to nondeterminism and the idea is to explain a parallel constructor $c \parallel c'$ in terms of the possible atomic actions of $c$ and $c'$ and the set of their interleavings [Plo83]. The original Egli-Milner notion of powerdomain works only for flat domains [Hrb87]. Plotkin showed how to define powerdomains for arbitrary cpo's [Plo76]. Unfortunately, Scott domains are not closed under the Plotkin powerdomain constructor and therefore cannot treat nondeterminism and parallelism adequately, some how.

A more general framework is the SFP domains (Sequence of Finite complete Partial orders), introduced by Plotkin to resolve the above mentioned difficulties. Basically, the SFP domains are cpo's obtained as limits of directed chains of finite cpo's.

It has been an open question how SFP domain can be represented as information system in a better way following Scott lines [Zha94]. A couple of attempts were made to find a natural extension of Scott's representation. However, none of these seems to provide a fully satisfactory treatment of SFP domains.

In this work, we define the Plotkin information systems and show that this type of information systems represent adequately SFP domains, in the sense that our representation is categorically equivalent to SFP domains.

Our representation is given with the intention of keeping as close as possible the spirit of Scott's original construction, in this way, a more concrete, logical account of SFP domains is provided. Construction such as Plotkin powerdomain and function space are given, as well as a cpo of Plotkin information systems to solve, using the simples technique of least fixed-point, the recursively defined SFP domain equations.

2 SFP domains

In this section we assume that the readers are familiar with the basic concepts of the domain theory and category theory. A good reference are [Plo83] and [AL91], respectively.

Definition 2.1 Let $D$ and $E$\(^1\) be cpo's. A continuous function $f : E \rightarrow D$ is a retraction if there is a continuous function $g : D \rightarrow E$ such that $f \circ g = \text{Id}_D$, in this case $D$ is a retract of $E$. If $g$ satisfies the further condition that $g \circ f \subseteq \text{Id}_E$ then $f$ is called a projection.

\(^1\)In the follow we write $D$ to indicate the cpo $D = (D, \leq_D, \perp_D)$. Analogously for the other cpo's $E, D_1$, etc.
Definition 2.2 Let $\mathcal{D} = \{D_1, \ldots, D_n, \ldots\}$ be a family of cpo's. We said that $\mathcal{D}$ is a directed sequence if for each $m$ and $n$, such that $m \leq n$, there exists a projection $f : D_n \to D_m$.

Definition 2.3 A cpo $D$ is a SFP domain if $D \cong \varprojlim \mathcal{D}$, i.e. $D$ is the inverse limit of the directed sequence of finite cpo’s $\mathcal{D}$.

A practical characterization of SFP domains is obtained in terms of minimal upper bounds.

Definition 2.4 Let $D$ be a cpo. An element $u \in D$ is a minimal upper bound of a subset $X \subseteq D$ if, and only if, it is an upper bound of $X$ which is not strictly greater than any other upper bound of $X$. We write $\text{MUB}(X)$ for the set of minimal upper bounds of $X$. $\text{MUB}(X)$ is said to be complete if whenever $u$ is an upper bound of $X$, $v \leq u$ for some $v \in \text{MUB}(X)$.

Proposition 2.5 [Plo76] Let $D$ be an algebraic cpo. If $X \subseteq \text{fin} D^0$ and $\text{MUB}(X)$ is complete then, $\text{MUB}(X) \subseteq D^0$.

For a subset $X$ of $D$, let $U^0(X) = X$ and $U^{i+1}(X) = \{u : u \in \text{MUB}(S) \text{ for some } S \subseteq \text{fin} U^i(X)\}$, for $i \geq 0$. It is convenient to write $U^i(X)$ instead of $\bigcup_{i \geq 0} U^i(X)$, that is, $U^i(X)$ is the least set containing $X$ and closed under $\text{MUB}$.

Theorem 2.6 [Plo76] $D$ is a SFP domain if, and only if, the following statement holds:

1) $D$ is $\omega$-algebraic,

2) $\forall X \subseteq \text{fin} D^0, \text{MUB}(X)$ is complete, and

3) $U^*(X)$ is finite.

Proposition 2.7 Let $D$ be an algebraic cpo such that $\forall S \subseteq \text{fin} D^0, \text{MUB}(S)$ is complete. If $X \subseteq \text{fin} D^0$ then, $U^*(X) \subseteq D^0$.

Proof: If $x \in U^*(X)$, then, by definition of $U^*(X)$, $x \in U^i(X)$ for some $i \in \mathbb{N}$. We will prove by induction on $i$ that $x \leq \bigsqcup X$.

If $i = 0$ then $x \in X$ and therefore $x \in D^0$. The inductive hypothesis is that if $x \in U^n(X)$ then, $x \in D^0$. So, if $i = n + 1$ and $x \in U^{n+1}(X)$, then, by definition, $x \in \text{MUB}(S)$ for some $S \subseteq \text{fin} U^n(X)$. So, by inductive hypothesis, $S \subseteq \text{fin} D^0$. Thus, by hypothesis, $\text{MUB}(S)$ is complete, then, by proposition 2.5, $\text{MUB}(S) \subseteq D^0$. Therefore, $x \in D^0$.

Observe that not always an $\omega$-algebraic cpo is a SFP domain. Three usual examples are given in figure 1.

The cpo of figure 1.a do not satisfy the condition 2 of theorem 2.6. The cpo's b and c in figure 1 do not satisfy the condition 3 of theorem 2.6. On the other hand, not always a SFP domain is a Scott domain, an example of this is the cpo of figure 2.

The category $\text{SDom}$, of Scott domains with continuous functions, is a full sub-category (proper) of the category $\text{SFP}$, which has SFP domains as object and continuous functions as morphisms, and the category $\text{SFP}$ is a full sub-category of the category $\text{Alg}$, that have $\omega$-algebraic cpo's as object and continuous functions as morphisms. In fact, $\text{SFP}$ is the largest full sub-category of $\text{Alg}$ closed under the function space constructor [Plo76] and this is confirmed by the Smyth theorem, who proved the following conjecture of Plotkin [Smy83].

\[\text{The notation } X \subseteq \text{fin} Y \text{ means that } X \text{ is a finite subset of } Y \]
Theorem 2.8 [Smy83] Let $D$ be an $\omega$-algebraic cpo. If $D \rightarrow D$ is a $\omega$-algebraic cpo then, $D$ and $D \rightarrow D$ are SFP domains.

Corollary 2.9 [Smy83] Let $C$ be a cartesian closed category. If $C \subseteq \text{Alg}$ then, $C \subseteq \text{SFP}$.

![Figure 1: algebraic cpo's which are not SFP domains](image1)

![Figure 2: Simple case of a cpo that not is Scott domain](image2)

3 SFP domains as Plotkin information systems

Algebraic cpo's are obtained as completion by ideals of a poset [Law87]. The elements of this poset are tokens of information on the elements of the algebraic cpo. Under this inspiration
D. Scott introduced the notion of Information System, an alternative approach to Scott domains \[\text{[Sco82b]}\] and \[\text{[LW84]}\]. We extend the original definition of information system in order to obtain as domain of elements an algebraic cpo instead of a Scott domain.

Roughly speaking, an information system is a triple compound by a basic information set (finite in the sense that they are informations of computations inferred from observing a program execution at a finite time with a finite amount of work \[\text{[Vic89]}\]), a consistency predicate and an entailment relation. Since we are concerned with the computational aspects of the information systems, they must support a finite notion of consistency. Finally the entailment relation must allow us to decide when we can deduce from a set (consistent) of information other informations.

**Definition 3.1** A **Pre-Information System** \((\text{PIS})\) is a triple \(\mathcal{I} = (I, \text{Con}, \rightarrow)\), where \(I\) is a countable information set, \(\text{Con}\) is a subset of \(\mathcal{P}^{\text{fin}}(I)\) (finite part of \(I\)), named consistency predicate, and \(\rightarrow\) is a subset of \(\text{Con} \times I\), named entailment relation, such that

I. \(\emptyset \in \text{Con}\)

II. If \(a \in I\) then, \(\{a\} \in \text{Con}\)

III. If \(X \vdash a\) then, \(X \cup \{a\} \in \text{Con}\)

IV. If \(a \in X\) and \(X \in \text{Con}\) then, \(X \vdash a\)

V. \(\forall X, Y \in \text{Con} \text{ and } a \in I, \text{ if } X \vdash Y^3 \text{ and } Y \vdash a\) then, \(X \vdash a\)

A PIS informs on the elements of a domain which can be identified with the set of information in the system which are satisfied by the element. Of course, such a set must be consistent and must contain all informations which can be deduced from it.

**Definition 3.2** The elements of a PIS \(\mathcal{I} = (I, \text{Con}, \rightarrow)\) are subsets \(x\) of \(I\) which satisfy the following conditions

a) **pre-finitely consistent**: If \(X \subseteq^{\text{fin}} x\) then, \(\exists Y \in \text{Con}\) such that \(X \subseteq Y \subseteq x\)

b) **\(\vdash\)-closed**: If \(X \subseteq x\) and \(X \vdash a\) then, \(a \in x\).

The set of element of a PIS \(\mathcal{I} = (I, \text{Con}, \vdash)\) is denoted by \(|\mathcal{I}|\).

**Lemma 3.3** Let \(\mathcal{I} = (I, \text{Con}, \vdash)\) be a PIS. If \(X \in \text{Con}\) then, \(\overline{X} \in |\mathcal{I}|\), where \(\overline{X} = \{a \in I : X \vdash a\}\).

**Proof:** If \(Y \subseteq^{\text{fin}} \overline{X}\) then, by definition of \(\overline{X}\), we have that \(X \vdash Y\). So, by the property III of definition 3.1, \(X \cup Y \in \text{Con}\). Therefore, since \(X \cup Y \subseteq \overline{X}\), \(\overline{X}\) is pre-finitely consistent. But, by definition, \(\overline{X}\) is \(\vdash\)-closed then, \(\overline{X} \in |\mathcal{I}|\).

**Theorem 3.4** If \(\mathcal{I} = (I, \text{Con}, \vdash)\) is a PIS then, \(|\mathcal{I}|, \subseteq\) is a \(\omega\)-algebraic cpo with \(\text{Con} = \{\overline{X} : X \in \text{Con}\}\) as the set of finite elements.

\(^3\)The notation \(X \vdash Y\) is a shorthand for \(\forall a \in Y, X \vdash a\).
PROOF: First, we will show that $\langle |I|, \subseteq \rangle$ is a cpo. In this sense, observe that $\emptyset \in Con$ and, from lemma 3.3, $\emptyset \in |I|$. So trivially, $\emptyset$ is the least element of the poset $\langle |I|, \subseteq \rangle$. On the other hand, if $\Delta \subseteq |I|$ is a directed set then, $\forall x \in \Delta, x \subseteq \cup_{y \in \Delta} y$. If $z \in |I|$ is another upper bound of $\Delta$, and $a \in \cup_{y \in \Delta} y$ then, $a \in y$ for some $y \in \Delta$. So, for upper boundness, $a \in z$ and therefore, $\cup_{y \in \Delta} y \subseteq z$. In this form, we showed that if $\cup_{y \in \Delta} y \in |I|$ then $\bigcup \Delta = \cup_{y \in \Delta} y$. Therefore, we must show that $\cup_{y \in \Delta} y \in |I|$ in order to prove that $\langle |I|, \subseteq \rangle$ is a cpo.

If $X = \{a_1, a_2, \ldots, a_n\} \subseteq \cup_{y \in \Delta} y$ then, there are $x_1, x_2, \ldots, x_n \in \Delta$ such that $a_i \in x_i$ for each $i = 1, \ldots, n$. Thus, for directness, there exist $x^* \in \Delta$ such that $X_i \subseteq x^*$, for each $i = 1, \ldots, n$. Therefore $X \subseteq x^*$, and by property a of definition 3.2, there exist $Y \in Con$ such that $X \subseteq Y \subseteq x^*$. So, $Y \subseteq \cup_{y \in \Delta} y$ and therefore, $\cup_{y \in \Delta} y$ is pre-finitely consistent.

If $X = \{a_1, a_2, \ldots, a_n\} \subseteq \cup_{y \in \Delta} y$ and $X \vdash a$ then, as above, there is a $x^* \in \Delta$ such that $X \subseteq x^*$. By property b of definition 3.2, $a \in x^*$. So, $a \in \cup_{y \in \Delta} y$ and therefore, $\cup_{y \in \Delta} y$ is $\vdash$-closed. Thus, $\cup_{y \in \Delta} y \in |I|$ and therefore, $\langle |I|, \subseteq \rangle$ is a cpo.

Let $X = \{a_1, a_2, \ldots, a_n\} \in Con$ and $\Delta \subseteq |I|$ be a directed set. If $X \subseteq \bigcup \Delta$ then, as above, $X \subseteq \cup_{y \in \Delta} y$. Since, trivially, $X \subseteq \cup_{y \in \Delta} y$, there exist $x^* \in \Delta$ such that $X \subseteq x^*$ and by $\vdash$-closedness of $x^*$, we have that $X \subseteq x^*$. Therefore, $X$ is finite. Let $x$ be a finite element. Clearly, $\Delta = \{X : X \in Con$ and $X \subseteq x\}$ is a directed set and $x = \cup_{y \in \Delta} y = \bigcup \Delta$. By finiteness of $x$, there is $y \in \Delta$ such that $x \subseteq y$. But, $y = \overline{X}$ for some $X \in Con$. Thus, $Con$ is the set of finite elements of $|I|$ and, therefore, $\langle |I|, \subseteq \rangle$ is a $\omega$-algebraic cpo.

**Definition 3.5** Let $I = \langle |I|, Con, \vdash \rangle$ be a PIS. $I$ is called a Plotkin Information System if $\forall Y^l \subseteq^f |I|, \exists X = \{X_1, \ldots, X_n\} \subseteq^f Con$ such that

a) If $Y^l \subseteq Y$ and $Y \in Con$ then, $Y^l \in X$,

b) $\forall Y^l \subseteq X$. If $Z \vdash X_j, \forall X_j \in X^l$ then, $\exists X_j \in X$ such that, $Z \vdash X_i$ and $X_i \vdash X_j \forall X_j \in X^l$.

In the following, we prove that all Plotkin information system define canonically a SFP domain.

**Theorem 3.6** If $I = \langle |I|, Con, \vdash \rangle$ is a Plotkin Information System then, $\langle |I|, \subseteq \rangle$ is a SFP domain.

**PROOF:** We will show that $\langle |I|, \subseteq \rangle$ satisfy the properties of theorem 2.6.

1. Trivially, from theorem 3.4, $\langle |I|, \subseteq \rangle$ is a $\omega$-algebraic cpo.

2. Let $\overline{X} \subseteq |I|$ be a subset of finite elements of $\langle |I|, \subseteq \rangle$ then, from theorem 3.4, $\overline{X} = \{Y_1, \ldots, Y_m\} \subseteq^f Con$. If $Y = \cup_{i=1}^m Y_i$ then, $Y \subseteq^f |I|$ and therefore there exist $X = \{X_1, \ldots, X_n\}$ satisfying the properties a) and b) of definition 3.5. Thus, trivially from condition a of definition 3.5, for each $Y_i \in X = \{Y_1, \ldots, Y_m\}$ we have that $Y_i \in X$ and therefore $Y \subseteq X$. On the other hand, if $x \in |I|$ is an upper bound of $\overline{Y}$ then, $Y \subseteq^f x$. So, by definition 3.2, there is $Z \in Con$ such that $Y \subseteq Z$ and therefore $Z \vdash Y_j$ for each $j = 1, \ldots, m$. Hence, by definition 3.5.b, there is $X_i \in X$, such that $Z \vdash X_i$ and $X_i \vdash Y_j$ for each $j = 1, \ldots, m$. So, $\overline{X} \subseteq \overline{Z} \subseteq X$. Therefore, $MUB(\overline{Y})$ is complete.

3. Let $\overline{Y} = \{Y_1, \ldots, Y_m\} \subseteq Con$ and $Y = \cup_{i=1}^m Y_i$. Thus, $Y \subseteq^f |I|$ and, therefore, there is $X = \{X_1, \ldots, X_n\} \in Con$ satisfying the conditions of definition 3.5. Hence, analogously to item b), $Y = \{Y_1, \ldots, Y_m\} \subseteq X$. Thus, if $x \in U^*(\overline{Y})$ then, by proposition 2.7, $x$ is finite, i.e. $x = \overline{X}$ and $X \in Con$. From definition of $U^*$, there is $i$ such that $x \in U^*(\overline{Y})$. Con proves that $\overline{Y}$ is finite.
We will prove, by induction on \( i \), that \( X^i \subseteq X \). If \( i = 0 \) then, by definition of \( U^0 \), \( X^0 \subseteq X \). Since \( Y \subseteq X \) then, \( X^i \subseteq X \). If \( i = p + 1 \) then, \( X^i \in MUB(S) \) for some \( S = \{ s_1, \ldots, s_k \} \subseteq fin \, U^p(Y) \). Thus, by inductive hypothesis, \( s_l \in X \) for each \( l = 1, \ldots, k \) and therefore \( s_l = X^i \) for some \( S \subseteq X \). Since \( X^i \) is a minimal upper bound for \( S \) then, \( X^i \vdash S_l \) for each \( l = 1, \ldots, k \). Thus, by condition \( b \) of definition 3.5, there is \( X_i \in X \) such that \( X_i \vdash S_l \) for each \( l = 1, \ldots, k \) and \( X^i \vdash X_i \). So, \( X_i \) is an upper bound for \( S \) and \( X_i \subseteq X^i \). But, since \( X^i \) is a minimal upper bound for \( S \) then, \( X^i = X_i \). Therefore, \( x = X^i \subseteq X \).

In the follow, we will prove that all SFP domain define, canonically, a Plotkin information system.

**Definition 3.7** Let \( D = (D, \subseteq, \bot) \) be a SFP domain. Define \( IS(D) = (D^0, \mathrm{Conv}_D, \vdash_D) \), where

1. \( X \in \mathrm{Conv}_D \) if, and only if, \( X \subseteq fin \, D^0 \) and \( \bigcup X \) exist.
2. \( X \vdash_D a \) if, and only if, \( X \in \mathrm{Conv}_D \) and \( a \subseteq \bigcup X \).

**Theorem 3.8** If \( D = (D, \subseteq, \bot) \) is a SFP domain then, \( IS(D) \) is a Plotkin Information System.

**Proof:** First, we will prove that \( IS(D) \) is a pre-information system.

I. Trivially, \( \emptyset \in \mathrm{Conv}_D \).

II. If \( a \in D^0 \) then, trivially, \( a \) is a least upper bound of \( a \). Therefore, \( \{ a \} \in \mathrm{Conv}_D \).

III. If \( X \vdash_D a \) then, \( a \subseteq \bigcup X \). So, \( \bigcup X = \bigcup (X \cup \{ a \}) \). Therefore, \( X \cup \{ a \} \in \mathrm{Conv}_D \).

IV. If \( a \in X \) and \( X \in \mathrm{Conv}_D \) then, by definition, \( a \subseteq \bigcup X \). Therefore, \( X \vdash_D a \).

V. Let \( X, Y \in \mathrm{Conv}_D \) and \( a \in D \) such that \( X \vdash_D Y \) and \( Y \vdash_D a \). By definition 3.7, \( b \subseteq \bigcup X \) for each \( b \in Y \) and, therefore, \( \bigcup Y \subseteq \bigcup X \). Since \( a \subseteq \bigcup Y \) then, \( a \subseteq \bigcup X \). Therefore, \( X \vdash_D a \).

On the other hand, let \( Y \subseteq fin \, D^0 \). We will show that \( IS(D) \) satisfy the property \( a) \) and \( b) \) of definition 3.5, where \( X = \{ X \in \mathrm{Conv}_D : X \subseteq U^*(Y) \} \). Notice that, by proposition 2.7, \( U^*(Y) \subseteq fin \, D^0 \).

a) If \( Y \subseteq Y \) and \( Y \in \mathrm{Conv}_D \) then, \( Y \subseteq U^0(Y) = Y \subseteq U^*(Y) \). Therefore, \( Y \in X \).

b) Let \( X \subseteq X \) and \( Z \in \mathrm{Conv}_D \), such that \( Z \vdash_D X \) for each \( X \in X \). Since each \( X \subseteq U^*(Y) \), \( \bigcup_{X \in X} X \subseteq U^*(Y) \) and, from definition of \( U^* \), \( MUB(\bigcup_{X \in X} X) \subseteq U^*(Y) \). From definition 3.7, \( \bigcup Z \) is an upper bound of \( \bigcup_{X \in X} X \). Thus, by definition of minimal upper bound, there exist \( b \in MUB(\bigcup_{X \in X} X) \) such that \( b \subseteq \bigcup Z \). Therefore, from definition 3.7, we have that \( Z \vdash_D \{ b \} \) and \( \{ b \} \vdash_D X \) for each \( X \in X \).
4 The category of Plotkin information systems

Plotkin information systems are equipped with morphism, called approximable mappings, which correspond to continuous functions between the associated SFP domains.

Definition 4.1 Let $I_1 = (I_1, \text{Con}_1, \vdash_1)$ and $I_2 = (I_2, \text{Con}_2, \vdash_2)$ be Plotkin Information Systems. An approximable mapping $\rho : I_1 \rightarrow I_2$ is a relation $\rho \subseteq \text{Con}_1 \times I_2$ such that:

- If $X \rho a_1 \land X \rho a_2 \land \cdots \land X \rho a_n$ then, $\{a_1, \ldots, a_n\} \in \text{Con}_2$
- If $X \rho a_1 \land X \rho a_2 \land \cdots \land X \rho a_n \land \{a_1, \ldots, a_n\} \vdash_2 a$ then, $X \rho a$
- If $X \vdash_1 Y \land Y \rho a$ then, $X \rho a$

Intuitively, an approximable mapping expresses how informations in one Plotkin information system entail informations of another one. Thus, an approximable mapping $\rho : I_1 \rightarrow I_2$ can be read as "if you are willing to give at least $X$ amount of information about the argument, then the mapping $\rho$ must to give at least the $a$ information about of the output value".

It is easy to see that Plotkin information systems with approximable mappings form a category, called PISys, in which mappings are composed by the usual composition of relations, the identity mapping on a Plotkin information system $I = (I, \text{Con}, \vdash)$ is the relation $\vdash$.

We have seen how a Plotkin information system $I$ represents a SFP domain $|I|$. In a similar way, an approximable mapping $\rho$ represents a continuous function $|\rho|$.

Proposition 4.2 Let $I_1 = (I_1, \text{Con}_1, \vdash_1)$ and $I_2 = (I_2, \text{Con}_2, \vdash_2)$ be Plotkin information systems. If $\rho : I_1 \rightarrow I_2$ is an approximable mapping then, $|\rho| : |I_1| \rightarrow |I_2|$, given by

$$|\rho| (x) = \{a : X \rho a \text{ for some } X \subseteq^\text{fin} x\}$$

is a continuous function between the SFP domains $(|I_1|, \subseteq)$ and $(|I_2|, \subseteq)$.

Thus, the operation $|\_|$ is a functor from the category PISys to the category SFP. On the other hand, to each SFP domain $D$ corresponds an associated Plotkin information system $IS(D)$. In a similar way a continuous function $f$ represents an approximable mapping, $IS_f$.

Proposition 4.3 Let $D$ and $E$ be SFP domains. If $f : D \rightarrow E$ is a continuous function then, the relation $IS_f \subseteq \text{Con}_D \times E^0$, defined by

$$X(IS_f)a \text{ if, and only if, } a \leq b \text{ for some } b \in \text{MUB}(X),$$

is an approximable mapping between the Plotkin information system $IS(D) = (D^0, \text{Con}_D, \vdash_D)$ and $IS(E) = (E^0, \text{Con}_E, \vdash_E)$.

Thus, $IS$ is a functor from the category SFP to the category PISys. Observe that for any SFP domain $D$ the function $\phi : D \rightarrow |IS(D)|$, given by $\phi(x) = \{y \in D^0 : y \leq x\}$, is a continuous bijection, which determines therefore that $D$ and $|IS(D)|$ are isomorphic orders. Thus, the functors $|\_| : \text{PISys} \rightarrow \text{SFP}$ and $IS : \text{SFP} \rightarrow \text{PISys}$ establish an equivalence between the categories PISys and SFP and therefore, from a category-theoretical point of view, they are essentially the same. Still, we have gained something. The rather abstract category of SFP domains with continuous functions is represented by the more concrete category of Plotkin information system and approximable mappings. We can work concretely, with basic set theory, to produce constructions on Plotkin information systems.
5 Some constructions on Plotkin information systems

In this section we give two constructors on Plotkin information systems. We focus on more complicated function spaces and Plotkin powerdomain constructors while leaving the simplest ones, such as sum, product and lifting, to the reader.

Definition 5.1 Let $I_1 = \langle I_1, \text{Con}_1, \vdash_1 \rangle$ and $I_2 = \langle I_2, \text{Con}_2, \vdash_2 \rangle$ be Plotkin information systems. Define their function space by $I_1 \rightarrow I_2 = \langle \text{Con}_1 \times I_2, \text{Con}, \vdash_2 \rangle$ where

- Let $X = \{(X_1, a_1), \ldots, (X_n, a_n)\} \subseteq^{-\text{fin}} \text{Con}_1 \times I_2$. $X \in \text{Con}$ if, and only if, $\forall F \subseteq \{1, \ldots, n\}, (\bigcup_{i \in F} X_i \in \text{Con}_1 \Rightarrow \{a_i : i \in F\} \in \text{Con}_2)$

- $(X_1, a_1), \ldots, (X_n, a_n) \vdash (X, a)$ if, and only if, $\{a_i : X \vdash_1 X_i\} \vdash_2 a$

Proposition 5.2 Let $I_1 = \langle I_1, \text{Con}_1, \vdash_1 \rangle$ and $I_2 = \langle I_1, \text{Con}_1, \vdash_1 \rangle$ be Plotkin information systems. Then, $I_1 \rightarrow I_2$ is a Plotkin information system.

Proof: First, we will show that each $\rho \in I_1 \rightarrow I_2$ is an approximable mapping between $I_1$ and $I_2$. To do so we have to prove that, if $\rho \in I_1 \rightarrow I_2 \subseteq \text{Con}_1 \times I_2$ then $\rho$ satisfies the properties of definition 4.1.

- Let $\{(X, a_i)\}_{i=1}^{n} \subseteq^{-\text{fin}} \rho$. Then, by construction, $X \in \text{Con}_1$ and, since $\rho$ is an element it is finitely consistent, i.e. $\{(X, a_i)\}_{i=1}^{n} \in \text{Con}_1$. Therefore, by definition 5.1, $\{a_i\}_{i=1}^{n} \in \text{Con}_2$.

- Let $\{(X, a_i)\}_{i=1}^{n} \subseteq^{-\text{fin}} \rho$ such that $\{a_i\}_{i=1}^{n} \vdash_2 a$. Trivially, by definition 5.1, $\{(X, a_i)\}_{i=1}^{n} \vdash (X, a)$. Since $\rho$ is an element it is $\vdash$-closed. Therefore, $(X, a) \in \rho$, i.e. $X \rho a$.

- Let $X \vdash_1 b$ for each $b \in Y$ and $(Y, a) \in \rho$. Since $\{(Y, a)\} \vdash (X, a)$ and $\rho$ is $\vdash$-closed, we have that $(X, a) \in \rho$, i.e. $X \rho a$.

Thus, by the proposition 4.2, the continuous function $\phi : I_1 \rightarrow I_2 \rightarrow I_1 \rightarrow I_2 \rightarrow I_2$ defined by $\phi(\rho) = |\rho|$ is trivially, by category equivalence, a bijection. Therefore, $|I_1| \rightarrow |I_2|$ is an isomorphic order of the SFP domain $|I_1| \rightarrow |I_2|$.

We will enrich the usual simple imperative language with a nondeterministic choice constructor, $c$ or $c'$. The idea is that an execution of $c$ or $c'$ consists of a choice and execution of $c$ or a choice and execution of $c'$. Let, $S$ be the countable set of states on which it is possible that the execution of an nondeterministic program start and finish. Thus, suppose that we want a denotational semantics, then we could expect

$$[c] : S \rightarrow \mathcal{P}(S)$$

In this way we use a nondeterministic function for our semantic. Thus, in order to give informations on the final states of a nonterministic program, we must give as informations a set
of basic informations on the states, i.e. if $I$ is the set of basic informations on $S$ then $P^\text{fin}_I$ is the set of basic informations on $P(S)$, since basic informations must be finite in the sense of [Vic89]. Thus, we must define an unary constructor on Plotkin information system which has as basic information the finite powerset of the original basic information and an adequately notion of consistency and entailment relations in order to interpret the different of nondeterminism. In our case we use the constructors of Plotkin powerdomain. Before we state the powerdomain definition, we need a little bit of terminology.

**Definition 5.4** Let $\mathcal{I} = \langle I, \text{Con}, \vdash \rangle$ be a Plotkin Information System. Define the pre-order of Plotkin on $\mathcal{I}$ as $\langle P^\text{fin}(I), \preceq_P \rangle$ where

$$A \preceq_P B \text{ if, and only if, } \forall a \in A \exists b \in B, \{a\} \vdash b \text{ and, } \forall b \in B \exists a \in A, \{a\} \vdash b.$$ 

**Definition 5.5** Let $\mathcal{I} = \langle I, \text{Con}, \vdash \rangle$ be a Plotkin information system. The Plotkin powerdomain of $\mathcal{I}$ is defined by $P^\mathcal{I} = \langle P^\mathcal{I}, \text{Con}^\mathcal{I}, \vdash^\mathcal{I} \rangle$ where

- $P^\mathcal{I} = P^\text{fin}(I)$
- $\overline{X} \in \text{Con}^\mathcal{I}$ if, and only if, $\overline{X} \subseteq P^\mathcal{I}$ and $\forall A \in \overline{X}, A \preceq_P B$, for some $B \in P^\mathcal{I}$
- $\overline{X} \vdash^\mathcal{I} A$ if, and only if, $\forall B \in \overline{X}, B \preceq_P C$ for some $C \in P^\mathcal{I}$, implies that $A \preceq_P C$.

**Proposition 5.6** Let $\mathcal{I} = \langle I, \text{Con}, \vdash \rangle$ be a Plotkin information system. Then, $P^\mathcal{I}$ is a Plotkin information system.

**Proposition 5.7** Let $\mathcal{I} = \langle I, \text{Con}, \vdash \rangle$ be a Plotkin information system. Then, $|P^\mathcal{I}|$ is an isomorphic order of the SFP domain $P_F(|\mathcal{I}|)$.

**Proof:** Observe that, from definition of powerdomains in [Plo76] and [Plo83], $P_F(|\mathcal{I}|)$ is obtained as the ideal completion of the pre-order $\langle P^\text{fin}(\text{Con}), \preceq_P \rangle$, where

$$\overline{X} \preceq_P \overline{Y} \iff \forall A \in \overline{X}, \exists B \in \overline{Y} \text{ such that } A \subseteq B \text{ and } \forall B \in \overline{Y}, \exists A \in \overline{X} \text{ such that } A \subseteq B.$$ 

Observe that, in order to show an isomorphism between $P_F(|\mathcal{I}|)$ and $|P^\mathcal{I}|$, it is sufficient to give a bijection function between the respective finite elements, i.e. from the main ideal in $\langle P^\text{fin}(\text{Con}), \preceq_P \rangle$ to $\text{Con}^\mathcal{I}$. Since each information system, and analogously each Plotkin information system, is “isomorphic” to an information system closed under conjunctions [Sco82b], we think that $\mathcal{I}$ is closed under conjunction. So, define $\phi : \text{Con}^\mathcal{I} \longrightarrow P_F(|\mathcal{I}|)^0$ and $\psi : P_F(|\mathcal{I}|)^0 \longrightarrow \text{Con}^\mathcal{I}$ by

$$\phi(\overline{X}) = \{ \{ A \} \}_{A \in \overline{X}} \text{ and } \psi(\{ \overline{X_1}, \ldots, \overline{X_n} \}) = \{ a_i \}_{i=1, \ldots, n}$$

where $a_i$ is the conjunction of $X_i$, i.e. $\{ a_i \} \vdash X_i$ and $X_i \vdash a_i$. So, clearly, $\phi \circ \psi = \text{Id}_{P_F(|\mathcal{I}|)^0}$ and $\psi \circ \phi = \text{Id}_{\text{Con}^\mathcal{I}}$. 

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$^4$ $P^\text{fin}_I$ is the set of the finite subsets of the set $I$, except the emptyset.

$^5$ A Plotkin information system $\mathcal{I} = \langle I, \text{Con}, \vdash \rangle$ is closed under conjunction if for each $X \in \text{Con}$ there is $a \in I$ such that $X \vdash a$ and $\{a\} \vdash X$.

$^6$ Let $D = (D, \leq_D)$ be a poset. $\downarrow x$ is the set $\{ y \in D : y \leq x \}$. 
6 A cpo of Plotkin information systems

Since we work with a concrete representation of SFP domains, we will replace the usual inverse limit constructions used in building up solutions to recursive domain equations by a fixed-point construction on a cpo of Plotkin information systems. The order on Plotkin information systems, \( \sqsubseteq \), captures an intuitive notion of subsystem of information as substructure of a Plotkin information system.

Definition 6.1 Let \( I_1 = (I_1, Con_1, \vdash_1) \) and \( I_2 = (I_2, Con_2, \vdash_2) \) be Plotkin information systems. We say that \( I_1 \) is a subsystem of \( I_2 \), denoted by \( I_1 \sqsubseteq I_2 \) if, and only if,

- \( X \in Con_1 \) if, and only if, \( X \subseteq I_1 \) and \( X \in Con_2 \)
- \( X \vdash_1 a \) if, and only if, \( X \subseteq I_1, a \in I_2 \) and \( X \vdash_2 a \).

Observe that, we require that Plotkin information systems have a denumerable set of information tokens. This restriction, allows that the class of all Plotkin information system, \( \text{PIS} \), to be a set (not denumerable) without danger to fall into paradox.

Lemma 6.2 Let \( \text{PIS} \) be the set of all Plotkin information systems. The relation \( \sqsubseteq \) is an order relation on \( \text{PIS} \), i.e. \( \langle \text{PIS}, \sqsubseteq \rangle \) is a poset.

Theorem 6.3 The poset \( \langle \text{PIS}, \sqsubseteq, 1 \rangle \) is a cpo, where \( 1 = (\emptyset, \{\emptyset\}, \emptyset) \).

Proof: Trivially 1 is a Plotkin information system and, for any other Plotkin information system \( I \), we have that \( 1 \sqsubseteq I \). Therefore 1 is the least element of the poset \( \langle \text{PIS}, \sqsubseteq \rangle \).

On the other hand, if \( \Delta \) is a directed set of the poset \( \langle \text{PIS}, \sqsubseteq \rangle \) then, \( I^* = (I^*, Con^*, \vdash^*) \) is the plotkin information system that is the least upper bound of \( \Delta \), where \( I^* = \bigcup_{I \in \Delta} I \), \( Con^* = \bigcup_{I \in \Delta} Con \) and \( \vdash^* = \bigcup_{I \in \Delta} \vdash^* \).

Let \( C \) be an unary operation on Plotkin information systems. We say that \( C \) is a continuous operation if it is a continuous function from the cpo \( \text{PIS} \) to itself.

Proposition 6.4 The unary operations \( F \) and \( P \) on \( \text{PIS} \), defined by \( F(I) = I \rightarrow I \) and \( P(I) = I^P \), are continuous.

Observe that by Tarki's theorem of least fixed-point, each continuous (operation) function has a least fixed-point. Thus, if \( F \) is a continuous operation on \( \text{PIS} \) then, there exist a Plotkin information system \( I \) such that

- \( I = F(I) \)
- If \( I' \) is any other Plotkin information system such that \( I' = F(I') \) then, \( I \sqsubseteq I' \).

This theorem allow us to solve recursive equation in a simples way. For example, if \( \oplus \) is a binary operation (continuous) and \( A \) is a fixed Plotkin information system representing an atomical data type then, we can solve, applying the theorem of least fixed-point, the recursive equations

\[ I = A \oplus (I \rightarrow I)^7 \text{ and } I = I^P. \]

\(^7\)We uses this equations instead of natural equation \( I = I \rightarrow I \) because that the last equation has as solution the trivial Plotkin information system 1
7 Conclusion

We believe that the results presented in this work strongly indicate that we closed, as much as possible, the question of how one can represent in the best way a SFP domain as an information system, following Scott's works. Several attempts were made in order to find an extension of Scott's representation. However, none of these seems to provide a fully satisfactory treatment of SFP domains.

The first of this tentative is the Scott manuscript [Sco82a] in which she succeed to give a representation only for 2/3 SFP domains. In [Abr91] pre-local domains are used to represent SFP domains, with the advantage that proposition of higher types can be constructed more naturally by allowing explicit conjunctions. However, that treatment is not near to the spirit of information systems, because allows the explicit use of logical operators and employs heavy machinery. The work of Droste and Göbel [DG90] gives an axiomatization of the domains determined by non-deterministic information systems. Since entailment, such as $\emptyset \vdash \emptyset$, are allowed we have that a domain of elements do not need to be a cpo. Finally, the work of [Zha89] and [Zha94], modify the original structure of information systems, eliminating the consistence predicate (it can be recovered from the entailment relation) and instead of using entailment of the form $X \vdash a$, he works with the classical Gentzen style $X \vdash Y$, providing explicit disjunction of propositions in $Y$. This approximation is better than the previous one, however is not completely faithful to the original structure of information system and it is much more complex than the one we give in this work.

We extend with success, in a simples way, the original notion of information systems and we prove that this extension is an adequate representation for SFP domains. In fact, in our representation, all Scott information systems are Plotkin information systems, without any transformation of the structure as in the other tentatives.

In this work, we show that the category of Plotkin information systems with approximable mappings, is equivalent to the category of SFP domains with continuous functions. From the point of view of categorical construction, Plotkin information systems say no more than SFP domains, they are just a representation of the old category of SFP domains. However, Plotkin information systems do have potentially more structure than SFP domains and quite likely the morphisms of them have a greatest amount of details. With new morphisms come new constructions, construction that might take into account the internal structure of tokens of information. Thus, properties of SFP domains can be derived rather than postulated and, the representation of information system makes them more amenable. This would be a strong point to close-up the connection between denotational semantics and the proof of properties of non-deterministic programs, and could explore the idea that the denotation of a programming constructor, like Plotkin powerdomain or function spaces, can be seen as the set of assertions on the constructors.

We show that the class of Plotkin information systems with the natural order of substructure is a cpo (not denumerable). Thus the standard SFP domain constructors can be made continuous on this cpo. So, the solution of recursive (SFP) domain equation reduces to the more familiar construction of forming the least fixed-point of a continuous function.
References


[Sco82a] Dana Scott. Notes on CPO’s/SFP objects and the like, 3 November 1982.


