Confluence of Term Rewriting Systems under Joinability of Critical Pairs in One Step of Parallel Reduction

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Abstract

As it is well-known, the critical pair lemma enables a finite test for confluence of (finite) terminating term rewriting systems. If termination does not hold then proving confluence is much more difficult. We consider the problem of verifying confluence of unconditional term rewriting systems without the assumption of termination. In particular, we examine the problem of confluence under the assumption that the critical pairs can be joined (either from left to right or from right to left) by one step of parallel reduction. When the one step parallel joinability of the critical pairs is from right to left one has an interesting question: the thirteenth open problem on rewriting proposed in [DJK91].
1 Introduction

In the field of computation theory, rewriting represents an appropriate model for functional languages. By imposing some fixed rewriting strategy, such as innermost reduction, one obtains a very close correspondence with the functional evaluation mechanism used in functional programming languages like LISP, ML, GOFER, etc. By admitting simultaneous reductions at disjoint positions of terms, one obtains an interesting theoretical model of parallel functional computation.

Termination and confluence are the key properties of rewriting systems. Many criteria for proving (and guaranteeing) termination have been developed, see for example [Ste94]. For terminating rewriting systems, finite criteria to guarantee confluence have been developed too. In fact, as it is well-known, the critical pair lemma enables a finite test for confluence of (finite) terminating term rewriting systems [KB70].

Also, these properties have been exhaustively examined for conditional rewriting systems (see for example [Aya94], [Gra94]), being probably the most important extension of purely equational rewriting, since they represent a model of computation for which, in the case that the conditions are constraints or predicates in a built-in theory, conditions that can be treated by a mechanism different from rewriting, the combination of the functional with another paradigms of computation, such as the logical one, arises naturally.

Without the assumption of termination, criteria to guarantee confluence are either very restrictive or lack of practical interest. One of the most studied types of not necessarily terminating rewriting systems is the one of the so-called orthogonal rewriting systems, i.e., rewriting systems which are left-linear and non-overlapping. It is well-known that any orthogonal rewriting system is confluent without being necessarily terminating. See [Klo92] for basic results about orthogonal rewriting systems.

Also, left-linear term rewriting systems are confluent if, for every critical pair \((P, Q)\) (where \(P = l_1[l_2]σ \leftarrow l_1[l_2]σ \rightarrow r_1σ = Q\), for some rules \(l_1 \rightarrow r_1\) and \(l_2 \rightarrow r_2\) and a substitution \(σ\)) we have \(P\) reduces in one parallel step to \(Q\).

In this work, we study the following question, proposed in [DJK91] and subsequently in [DJK93] as an open problem:

\[
\text{Is confluent a left-linear term rewriting system for which for every critical pair } (P, Q), Q \text{ reduces in one parallel step to } P? 
\]

We study, in detail, the beautiful inductive Huet's proof of the confluence for left-linear term rewriting systems, when for all critical pairs one has joinability in one step of parallel reduction from left to right (originally presented in [Hue80]) and explore the previous open question.

We show that, when the parallel joinability of the critical pairs is from right to left, confluence for the parallel rewrite relation holds; specifically, confluence in two steps of parallel reduction. Unfortunately, this result cannot be used to show confluence as it can be verified with simple counterexamples.

An abstract of this work appears in [Aya95].
2 Background

We use notations that are consistent with the standard ones in the field of rewriting (see [Dj90], [Klo92]). We recall the basic concepts and notations on rewriting used in this work.

Let $\mathcal{V}$ be a countably infinite set of variables and $\mathcal{F}$ be a set of function symbols with $\mathcal{V} \cap \mathcal{F} = \emptyset$. Associated to every $f \in \mathcal{F}$ is a natural number denoting its arity. A set of functions, $\mathcal{F}$, with its corresponding arities is usually called a signature. The set $T(\mathcal{F}, \mathcal{V})$ of terms over $\mathcal{F}$ and $\mathcal{V}$ is the smallest set which satisfies that $\mathcal{V} \subseteq T(\mathcal{F}, \mathcal{V})$ and if $f \in \mathcal{F}$ has arity $n$ and $t_1, \ldots, t_n$ are terms in $T(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{V})$. Some function symbols are allowed to be variadic then the definition of $T(\mathcal{F}, \mathcal{V})$ is generalized in an obvious way.

The length of a term $t$, $\lambda(t)$, is defined by: $\lambda(x) = 1$, if $x$ is a variable; $\lambda(f(t_1, \ldots, t_n)) = 1 + \sum_{i=1}^{n} \lambda(t_i)$, if $f$ is an $n$-ary function and $t_1, \ldots, t_n$ its arguments.

The set of variables occurring in a term $t$ is denoted by $V(t)$. Positions of a term consist of sequences of natural numbers and they are compared by the usual lexicographical ordering (which we shall denote by $\leq$). The topmost position of a term, i.e., of its root symbol, is denoted by $\epsilon$, the ‘empty’ string. Two incomparable positions $\pi$ and $\pi'$ are said to be parallel or disjoint. If $\pi \leq \pi'$ we say that $\pi$ is above $\pi'$ or that $\pi'$ is below $\pi$. The set of all positions of a term $t$ is denoted by $O(t)$. Let $\Phi$ and $\Psi$ be sets of positions of a given term. $\Phi$ is said to be dominant w.r.t. $\Psi$ if for every $\psi \in \Psi$ there is a position $\phi \in \Phi$ such that $\phi \leq \psi$. Also, we say that terms at positions $\Phi$ are dominant w.r.t. terms at positions $\Psi$. Concatenation of positions is denoted by juxtaposition. If $\pi$ is a position and $\Pi$ a set of positions then $\pi\Pi$ denotes the set of positions $\{\pi \rho \mid \rho \in \Pi\}$ and $\Pi\pi$ the set $\{\rho\pi \mid \rho \in \Pi\}$. The subterm of $t$ at position $\pi \in O(t)$ is denoted by $t[\pi]$. If $s = t[\pi]$ then $s$ is called the occurrence of $t$ at position $\pi$. Often $\pi$ is used to denote an arbitrary term related indirectly with the occurrence $t[\pi]$ and they should be not confused. The result of replacing in $t$ the subterm at position $\pi$ by $s$ is denoted by $t[\pi \leftarrow s]$ or simply by $t[s]_\pi$. $t[s]_\pi$ is also used to remark that $s$ is the subterm of $t$ at position $\pi$. If $\Pi$ is a subset of disjunct positions at $t$, then $t[s]_\Pi$ denotes the term obtained from $t$ by replacing all occurrences at positions in $\Pi$ with $s$. Also $t[s]_\Pi$ is used to remark that all occurrences of $t$ at positions in $\Pi$ are $s$. Let $\Pi = \{\pi_1, \ldots, \pi_n\}$ be a set of disjunct positions at $s$ and $\{t_1, \ldots, t_n\}$ be a set of terms; $s[t_1]_{\pi_1} \ldots [t_n]_{\pi_n}$ denotes the term $s[t_1]_{\pi_1} \ldots [t_n]_{\pi_n}$. Let $\rho, \pi, \gamma$ be positions such that $\rho = \pi\gamma$. This is extended in the obvious way to sets of positions: let $\Pi, \Gamma$ be sets of positions and $\pi$ a position such that $\pi\gamma \in \Pi$ if and only if $\gamma \in \Gamma$, then $\Pi\\pi = \Gamma$.

A substitution $\sigma$ is a mapping from $\mathcal{V}$ to $T(\mathcal{F}, \mathcal{V})$ such that its domain, $dom(\sigma) = \{x \in \mathcal{V} \mid x\sigma \neq x\}$, is finite. The homomorphic extension of a substitution $\sigma$ to a mapping from $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ is also denoted by $\sigma$. We assume that unification concepts (i.e., unifier and most general unifier) are known.

A term rewriting system (TRS for short) is a pair $(R, \mathcal{F})$ consisting of a signature $\mathcal{F}$ and a set $R \subseteq T(\mathcal{F}, \mathcal{V}) \times T(\mathcal{F}, \mathcal{V})$ of (rewrite) rules $l \rightarrow r$ with $l \notin \mathcal{V}$ and $V(r) \subset V(l)$. Instead of $(R, \mathcal{F})$ we write $R$ when $\mathcal{F}$ is clear from the context.

Given a TRS $R$, the rewrite relation $\rightarrow_R$ for terms $s, t \in T(\mathcal{F}, \mathcal{V})$ is defined as follows: $s \rightarrow_R t$ if there exists a rule $l \rightarrow r$ in $R$ by rewriting $s$, such that $s[l] = tr$ and $t = s[\sigma]_r$. We also write $\rightarrow$ when $R$ is clear form the context. The symmetric, transitive, reflexive and transitive-reflexive closures of $\rightarrow$ are denoted by $\leftrightarrow$, $\rightarrow^+$, $\rightarrow^*$ and $\rightarrow^\ast$, respectively. Analogously, $\leftrightarrow^\ast$ denotes the symmetric reflexive transitive
closure of $\rightarrow$. By $s \rightarrow^m t$ we mean that $s$ is reduced to $t$ in $m$ steps. Accordingly, $s \rightarrow^{\leq m} t$ means that $s \rightarrow^m t$ for some $m \leq n$. Note that $\rightarrow^=$ and $\rightarrow^{\leq 1}$ coincide. The **parallel rewrite relation** is defined from $\rightarrow$ as follows: let $s_1, \ldots, s_n$ be $n$ occurrences, for $n \geq 1$, of a given term $s$ at disjunct positions $\pi_1, \ldots, \pi_n$, respectively. If each disjunct occurrence $s_i$ rewrites into a term $t_i$, i.e., $s_i \rightarrow t_i$ for $i = 1, \ldots, n$, then it is said that $s$ rewrites in parallel to the term $t = s[t_1]_{\pi_1} \cdots [t_n]_{\pi_n}$. This relation is denoted by $s \parallel \rightarrow t$.

Two terms $s, t$ are **joinable** in $R$, denoted by $s \downarrow t$, if there exists a term $u$ with $s \rightarrow^* u \rightarrow^* t$, where $\leftrightarrow$ denotes the inverse of $\rightarrow$ and $\rightarrow^*$ its transitive reflexive closure. A term $s$ is irreducible or is a normal form if there is no term $t$ with $s \rightarrow t$. If $s \rightarrow^* t$ then $t$ is said to be a reduct of $s$.

A TRS $R$ is **terminating** or strongly normalizing if $\rightarrow$ is noetherian, i.e. if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow \cdots$. A TRS $R$ is **confluent** or has the Church-Rosser property (CR) if $(\leftrightarrow \circ \rightarrow) \subseteq (\rightarrow^* \circ \leftrightarrow)$, where $\circ$ denotes the relation composition. $R$ is said to be weakly Church-Rosser or **locally confluent** if $(\leftrightarrow \circ \rightarrow) \subseteq (\rightarrow^* \circ \leftrightarrow)$. $R$ is said to be strongly confluent (SCF) if $(\leftrightarrow \circ \rightarrow) \subseteq (\rightarrow^* \circ = \leftrightarrow)$. A confluent and terminating TRS is called convergent or complete.

If $l_1 \rightarrow r_1, l_2 \rightarrow r_2$ are rules of a TRS $R$, $\pi \in O(l_1)$ and $l_1|_{\pi}$ and $l_2$ are unifiable with most general unifier $\sigma$, then the ordered pair of terms $((l_1[r_2]_{\pi})\sigma, r_1\sigma)$ is said to be a critical pair of $R$ (obtained by overlapping $l_2 \rightarrow r_2$ with $l_1 \rightarrow r_1$ at position $\pi$).

It is well-known that for TRSs local confluence is equivalent to joinability of all critical pairs. This result is know as the critical pair lemma, which, originally, was proved by Knuth-Bendix using termination hypothesis [KB70]. Subsequently, Huet obtained the final version of the lemma without termination hypothesis [Hue80].

A TRS $R$ is said to be **non-overlapping** if there are no critical pairs between rules of $R$. It is **left-linear** (LL) if every variable occurs at most once in every left hand side of all rules of $R$. A left-linear and non-overlapping TRS, $R$, is said to be orthogonal. If every critical pair of a TRS, $R$, is obtained by an overlay, i.e. by overlapping left hand sides of rules at top positions then $R$ is said to be an overlay system. It is well-known that orthogonal TRSs are confluent (without being necessarily terminating). The first direct proof of this fact was presented by Rosen [Ros73].

## 3 Confluence of LL parallel closed systems

The purpose of this section is to examine, in detail, Huet's original proof of confluence for LL non terminating rewriting systems, whose critical pairs are joinable, from left to right, in one step of parallel rewriting. The result on confluence is presented as corollary of the following lemmas.

**Lemma 3.1 ([Hue80])** Let $\rightarrow$ be a SCF rewrite relation. Then $\rightarrow$ is CR.

**Proof:** We prove the lemma by double induction on $m$ and $n$ for all $u, v, w$ such that $w \rightarrow^m u \rightarrow^n v$.

If $n = 1$ then the proof is by induction on $m$:

If $m = 1$ then it holds by strongly confluence hypothesis.

Suppose it holds for $m$ and let $u, v, w, w'$ be terms such that $w \rightarrow^m u \leftrightarrow v$. By strongly confluence hypothesis there exists some $u'$ such that $w' \rightarrow^m u' \leftrightarrow v$, see part A of figure 1.
Figure 1: Strongly confluence implies confluence

Now, by induction hypothesis, there exists some \( u'' \) such that \( w \rightarrow^* u'' \leftarrow u' \). This proves that for all \( u, v, w \) such that \( w \leftarrow u \rightarrow v \), \( w \downarrow v \).

Now, suppose that this holds for \( n \) and let \( u, v, v' \), \( w \) be terms such that \( v'' \leftarrow v' \leftarrow u \rightarrow^* w \).

By induction hypothesis, there exists some \( u' \) such that \( v' \rightarrow^* u' \leftarrow w \) and by induction hypothesis, there exists some \( u'' \) such that \( v \rightarrow^* u'' \leftarrow u' \), see part B of figure 1.

**Lemma 3.2** Let \( \rightarrow \) be a rewrite relation and \( \downarrow \) its corresponding parallel rewrite relation. Then \( \rightarrow^* = \downarrow^* \).

**Proof:** Evidently, if \( u \rightarrow^* v \) then \( u \downarrow v \). Conversely, \( \downarrow^* \subseteq \rightarrow^* \) can be proved, by induction on \( n \), where \( u \downarrow^* v \) and \( \downarrow^* \) denotes \( n \) steps of parallel reduction.

If \( u \downarrow 1 v \), by reduction of the disjunct occurrences at positions \( \pi_1, \ldots, \pi_k \) of \( u \) with respective rules and substitutions \( l_i \rightarrow r_i \) and \( \sigma_i \), for \( i = 1, \ldots, k \), used in the one-step of parallel reduction, one has that \( u \rightarrow u[l_1 \sigma_1]_{\pi_1} \rightarrow \cdots \rightarrow u[l_1 \sigma_1]_{\pi_1} \cdots [l_k \sigma_k]_{\pi_k} \), i.e., \( u \rightarrow^* v \). Suppose that this holds for \( n \), i.e., if \( u \downarrow^n v \) then \( u \rightarrow^* v \), and let \( u, v, v' \) be terms such that \( u \downarrow^n v' \downarrow^* v \). Then, by induction hypothesis \( u \rightarrow^* v' \) and repeating the first step of the inductive proof \( v' \rightarrow^* v \), which completes the proof.

The following definition abbreviates simply what we previously called joinability in one step of parallel reduction from left to right.

**Definition 3.1** Let \( \rightarrow \) be a rewrite relation. \( \rightarrow \) is called (strongly) parallel closed if for all its critical pairs \((P, Q)\), \( P \downarrow Q \).

**Lemma 3.3** ([Hue80]) If \( \rightarrow \) is an LL and parallel closed rewrite relation, then \( \downarrow \) is SCF.

**Proof:** Let \( u, v, w \) be terms such that \( v \downarrow u \downarrow w \) by rewriting of subterms at positions \( \Pi \) and \( \Gamma \) of \( u \), respectively. We divide these sets of positions as follows:
\[ \Psi = \{ \pi \in \Pi \mid \exists \gamma \in \Gamma, \gamma \leq \pi \} \cup \{ \gamma \in \Gamma \mid \exists \pi \in \Pi, \pi \leq \gamma \}; \]
\[ \Phi = ((\Pi \cup \Gamma) \setminus \Psi) \cup (\Pi \cap \Gamma). \]

Note that \( \Phi \) is a set of dominant positions w.r.t. \( \Pi \cap \Gamma \) and \( \Psi \) can be considered as the corresponding set of dominated positions.

First of all, we prove that for all subterms \( u_\phi \) of \( u \) at positions \( \phi \in \Phi \), \( v_\phi \rightarrow t_\phi \rightarrow w_\phi \), for some term \( t_\phi \). Secondly, since the terms at positions \( \phi \in \Phi \) are dominant w.r.t. \( \Pi \cup \Gamma \), we conclude that \( v \rightarrow^* u[t_\phi]_{\phi \in \Phi} \rightarrow^* w \).

Firstly, suppose, w.l.o.g., that \( \phi \in \Pi \) and define \( \Gamma_\phi = \{ \gamma \in \Gamma \mid \gamma \geq \phi \} \). By rewriting in parallel all the occurrences \( u_\gamma \) for \( \gamma \in \Gamma_\phi \), one obtains \( u_\phi \rightarrow w_\phi \). On the other side, note that \( u_\phi \rightarrow v_\phi \), by applying some rule \( l \rightarrow r \) with substitution \( \sigma \), i.e., \( u_\phi = l\sigma \) and \( v_\phi = r\sigma \).

The proof is by induction on the addition of lengths of all terms at dominated positions, \( \Lambda(u, \Pi, \Gamma) \), defined by \( \Sigma_{\phi \in \Phi} \lambda(u_\phi) \).

Suppose that there are no non-variable occurrences of the rule \( l \rightarrow r \) in \( \Gamma_\phi \), i.e., for all \( \gamma \in \Gamma_\phi \), \( l|_{\gamma} \phi \) is either a variable or not defined (that is, either \( l|_{\gamma} \phi = x \) for some variable \( x \) or \( \gamma \phi \not\in O(l) \)).

Since \( \rightarrow \) is \( LL_\lambda \), one can define, from \( \sigma \), a new substitution \( \sigma' \), such that \( w_\phi = l\sigma' \). By rewriting \( r\sigma \) at all corresponding variable positions as is made in the parallel reduction from \( l\sigma \) into \( l\sigma' \), one obtains \( v_\phi = r\sigma \rightarrow r\sigma' \). Additionally, \( w_\phi = l\sigma' \rightarrow r\sigma' \). See figure 2.

Let \( t_\phi \) denote the term \( r\sigma' \).

![Figure 2: Only variable occurrences](image-url)

Now, suppose that there exists some non-variable position \( \gamma \in \Gamma_\phi \), i.e., \( l|_{\gamma} \phi \) is a non-variable term. Suppose that \( u_\gamma \) rewrites into \( w_\gamma \), by applying some rule \( l_1 \rightarrow r_1 \) with substitution \( \sigma_1 \). There exist a critical pair (from rules \( l \rightarrow r \) and \( l_1 \rightarrow r_1 \)), \( \langle P, Q \rangle \), and a substitution \( \rho \), such that \( P\rho = (u[w_\gamma]_\gamma)_\phi = l\sigma[r_1\sigma_1]_\gamma \phi \) and \( Q\rho = v_\phi = r\sigma \).

By parallel closed hypothesis \( P\rho \rightarrow^* Q\rho \). See top of figure 3.

Now, observe that \( P\rho \rightarrow^* w_\phi \) by rewriting all the remaining occurrences of \( w_\phi \) at positions \( \Gamma_\phi \setminus \{ \gamma \} \). We define, \( \Pi' \) as the set of occurrences in the reduction of \( P\rho \) into \( Q\rho \) and \( \Gamma' \) as the set of occurrences in \( (\Gamma_\phi \setminus \{ \gamma \}) \phi \) used in the parallel reduction to obtain \( w_\phi \). \( \Psi' \) and \( \Phi' \) are defined from \( \Pi' \) and \( \Gamma' \) as \( \Psi \) and \( \Phi \) were above defined. Evidently, \( \Lambda(P\rho, \Pi', \Gamma') <
Figure 3: Non-variable occurrences

\[ \Lambda(u, \Pi, \Gamma). \] By induction hypothesis, one obtains a term \( t_\phi \) such that \( v|_\phi \leftrightarrow t_\phi \leftrightarrow w|_\phi \). See bottom of figure 3.

Secondly, since the occurrences in \( \Phi \) dominate all occurrences in \( \Pi \cup \Gamma \), one can build a term \( t \) from \( u \), using all \( t_\phi \)'s, such that \( v \leftrightarrow t \leftrightarrow w \). In fact, \( t = u[t_\phi]|_{\phi \in \Phi} \).

This completes the proof.

Corollary 3.1 If \( \rightarrow \) is an LL and parallel closed rewrite relation then \( \rightarrow \) is CR.

Proof: By previous lemma \( \leftrightarrow \) is SCF. This implies, by lemma 3.1, that \( \leftrightarrow \) is CR and, by lemma 3.2, one concludes that \( \rightarrow \) is CR too.

4 Local confluence of LL inversely parallel closed systems

For the case of LL TRSs whose critical pairs are joinable in one step of parallel reduction from right to left, we prove a special type of local confluence for the parallel rewrite relation. This result does not imply the confluence of the rewrite relation as can be verified with a simple counterexample.

The following definition abbreviates what we have called critical pair joinability in one step of parallel reduction from right to left.

Definition 4.1 Let \( \rightarrow \) be a rewrite relation. \( \rightarrow \) is called inversely (strongly) parallel closed if for all its critical pairs \( (P, Q) \); \( Q \leftrightarrow P \).

Definition 4.2 \( \rightarrow \) is called 2-locally confluent if \((\leftarrow \circ \rightarrow) \subseteq (\leftrightarrow \circ \leftrightarrow \leftarrow)\).
Lemma 4.1 If → is an LL and inversely parallel closed rewrite relation, then its corresponding parallel relation, ↔, is 2-locally confluent.

Proof: Let u, v, w be terms such that v ↔ u ↔+ w, by rewriting of subterms at positions Π and Γ, respectively. We divide the sets of positions in dominant and dominated sets Ψ and Φ, which are defined as in the proof of lemma 3.3.

We show that for any position φ ∈ Φ and the sets of corresponding positions Ψφ = {ψ ∈ Ψ | ψ ≥ φ}, the terms v|φ and w|φ can be joined in at most two parallel rewriting steps.

The proof is divided into cases, as follows.

First case: suppose that φ ∈ Π and Ψφ = ∅. Then, obviously, u|φ = w|φ → v|φ.

Second case: suppose that φ ∈ Π, Ψφ ≠ ∅ and there is only variable positions in Ψφ (that is, if l → r and σ are the rule and corresponding substitution used to rewrite u|φ into v|φ and for all the positions ψ ∈ Ψφ, either l|ψ\|φ makes no sense or l|ψ\|φ is a variable). Then, by left-linearity hypothesis, one can define a new substitution σ' by rewriting at every variable position into u|φ, obtaining the following: u|φ = lσ ↔ lσ' = w|φ → rσ' and v|φ = rσ ↔ rσ', by rewriting (in parallel) at every corresponding variable position. See figure 4.

Third case: suppose that φ ∈ Π, Ψφ ≠ ∅ and there is non-variable positions in Ψφ.

Let ψ ∈ Ψφ be a non-variable position (that means: l|ψ\|φ is a non-variable subterm).

Then there is an instance (Pρ, Qρ) of a critical pair ⟨P, Q⟩, for a substitution ρ, such that Qρ = v|φ and Pρ = (u[w|ψ|φ])|φ. By hypothesis on critical pairs and consistency under substitutions, one has that Qρ ↔+ Pρ, i.e., (u[w|ψ|φ])|φ ↔+ v|φ. See figure 5.

Then one has that v|φ ↔+ w|φ.

The corresponding three cases for φ ∈ Γ are symmetric. Then, since Φ dominates all the redex positions in u, one obtains that v ↔+2 t ↔+ w, for some term t.

The general situation, combining the six possible cases, is illustrated in the figure 6.
Now, we illustrate the problems which arise when you attempt to follow a proof analogous to corollary 3.1 in order to obtain the confluence of rewrite relations under the hypothesis of left-linearity and the property of being inversely parallel closed. Under these hypotheses we don’t have that a rewrite relation is strongly confluent. Consequently, if you can either state a lemma of the form “2-local confluent rewrite relations are confluent” or show that this does not hold, then you obtain a proof of the analogous corollary or a way to generate a counterexample, respectively. Attempts to combine the previous lemma with a lemma similar to the lemma 3.1 fail, because 2-locally confluent systems are not necessarily confluent, as it can be illustrated with the following example.

Example 4.1 Consider the following rewrite relation over naturals (with zero).

\[
\begin{align*}
0 &\rightarrow 1 & 2 &\rightarrow 4 & n &\rightarrow n + 3, & n \geq 3 \\
0 &\rightarrow 2 & 2 &\rightarrow 5 & n &\rightarrow n + 2, & n \geq 3, n \equiv 1 \pmod{3} \text{ and } n \text{ even} \\
1 &\rightarrow 3 & & & n &\rightarrow n + 4, & n \geq 3, n \equiv 1 \pmod{3} \text{ and } n \text{ odd}
\end{align*}
\]

Obviously, the rewrite relation \( \rightarrow \) is LL. To verify that \( \rightarrow \) is 2-locally confluent but not convergent, observe the figure 7. This example is not completely satisfactory, because it is infinite. However, it is possible to build a finite semi-Thue system with similar properties, as follows\(^1\):

\(^1\)This rewriting system is obtained from the first one as is made for translating non-restricted grammars into context-sensitive grammars in [HU79]
Note that since parallel confluence and confluence are equivalent (this is a direct conclusion of lemma 3.2), the problem of searching for a counterexample can be reduced to build an LL inversely parallel closed non parallelly confluent rewriting relation.

In order to construct a counterexample, it is relevant to determine, which properties it should satisfy. Suppose there is a counterexample "→". Then → should be non-terminating (in other case the critical pair lemma applies implying the confluence) and with critical pairs (because orthogonal systems are confluent). Additionally, there should be non-overlaying critical pairs, as show the following simple lemma.

**Lemma 4.2** Let → be an LL, inversely parallel closed and overlay rewrite relation. Then → is CR.

**Proof:** Simply, note that → is parallel closed, because for each critical pair \( \langle P, Q \rangle \) resulting from an overlapping at root position \( \langle Q, P \rangle \) is also a critical pair.
Also, Toyama proved a mild version which states that overlay LL TRSs are confluent if for every critical pair \((P, Q)\), \(P \not\rightarrow^* Q\). See [Toy88].

Reviewing the structure of the proof of lemma 4.1, one can observe that there is another more simple question to answer: "Are LL TRSs for whose critical pairs \((P, Q)\), \(Q \rightarrow^*_1 P\) holds, confluent?" Obviously a positive answer to the main open problem implies a positive answer to the previous one.

5 conclusion

We conclude that, on the one side, a possible counterexample for the open problem should be non-terminating with non-overlaying critical pairs and, on the other side, if one expects to show the confluence of the rewrite relation, it is necessary to develop new approaches, which combine structural analysis over terms for inversely parallel closed 2-locally confluent parallel relations.

Our research, on this very simple stated open question, contributes sharing some light to understand its real complexity, without conjecturing about either a positive or negative answer.

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