Analysis of teaching strategies and learning challenges in Linear Algebra

Alethia Jiménez Piñón
Universidad Autónoma de Querétaro, Facultad de Informática,
Querétaro, México, 76230,
alethia463@gmail.com

and

Diana-Margarita Córdova-Esparza
Universidad Autónoma de Querétaro, Facultad de Informática,
Querétaro, México, 76230,
diana.cordova@uaq.mx

Abstract

This paper provides a systematic review focused on diagnosing learning difficulties and implementing teaching strategies in the context of linear algebra. The research aims to deepen the understanding of this topic over the last decade. The study, guided by four questions, analyzed 84 articles and ultimately included 41 in the review. The search strategy was based on the PRISM protocol, and specific indicators were used. The findings indicate that most authors in the review primarily use the APOE theory and genetic decomposition for formal diagnosis of learning problems. This approach helps build knowledge frameworks, especially in vector spaces and linear transformations. A key finding is the prevalent use of digital technology in both the models and strategies proposed in these studies. This review highlights opportunities for future research in diagnosing learning problems and developing innovative, technology-integrated strategies in education.

Keywords: Education, Didactic Strategy, Linear Algebra.

1 Introduction

Linear algebra is a fundamental branch of mathematics that plays a crucial role in various fields such as engineering, physics, computer science, and others. However, many students face significant challenges when learning linear algebra, often finding it abstract and conceptually difficult. Therefore, effective teaching strategies are necessary to help students better understand and retain the concepts.

The analysis of linear algebra teaching strategies involves exploring different pedagogical approaches [1, 2] to identify those that enhance student comprehension and participation. In the literature, methods such as interactive workshops, collaborative group work, and the use of technological tools such as computer algebra systems and online simulations are being examined. Each strategy has its strengths and potential drawbacks, and their effectiveness can vary depending on the learning environment and the individual needs of students.

Most of the technologies used focus on software applications for various topics in linear algebra. GeoGebra is one of the software tools used in research, along with platforms such as Moodle, e-Portfolios, interactive forums, videos, and WhatsApp groups. More recent studies have begun to explore artificial intelligence as a tool in the teaching of linear algebra, such as the use of ChatGPT [3]. Among the main conclusions of the study, it was determined that AI technology is not sufficient to replace the teacher; however, it is useful in supporting the student's learning process.

Teaching and learning mathematics often presents significant challenges for teachers. These challenges include covering the subject's content within the allotted time and addressing the diverse learning difficulties that students face. In addition, teachers must develop effective teaching strategies to enhance learning outcomes in mathematics.

Each researcher in this field brings their unique perspective, knowledge, and experience to analyze the state of knowledge about teaching and learning mathematics. Despite these efforts, learning problems persist in linear algebra [4], especially in abstract topics such as vector spaces and linear transformations [5].

To adapt teaching strategies for reproducibility and consistency, it is essential to establish a structured framework that minimizes subjectivity and adapts to diverse educational contexts. Research-based approaches, such as active learning methods, provide a foundation that is more likely to succeed in different contexts. Teaching strategies from primary to university level must be grounded in empirically proven learning theories. This ensures a foundation for methods that are validated both qualitatively and quantitatively.

Among the primary active methodologies identified in the state-of-the-art in mathematics teaching are gamification and problem-based learning. Gamification is a resource often used due to the diversity of topics it can address while motivating students in the subject of linear algebra. In the study presented by Wikie and Mytnik [6], it is emphasized that gamification is a tool that allows high participation in tasks that were previously considered difficult or uninteresting. It works exceptionally well in education by helping students develop learning habits and providing a visual representation of their progress in acquiring knowledge and skills. The article developed by [7] examines the integration of games into undergraduate mathematics education, focusing on students' perspectives regarding the use of quiz games after instruction. Specifically, it investigates students' views on a linear algebra quiz game used as a supplementary learning tool. A survey conducted with 78 students assessed their perceptions of the game's effectiveness in preparing for linear algebra exams. Participants reported high levels of participation and interest, describing the game as pleasant, dynamic, exciting, and helpful for reviewing key concepts. The results suggest that quiz games can serve as a valuable complement to mathematics instruction, enhancing student engagement and providing an enjoyable and effective way to reinforce course material.

The findings of the work of [8] suggest that incorporating active learning strategies in mathematics education can lead to more effective teaching and learning practices. These strategies not only foster a deeper understanding of mathematical concepts but also promote essential skills such as collaboration, communication, and critical thinking. The development of active learning environments leads to a more meaningful mathematics learning experience.

The work developed by [9] shows evidence that incorporating problem-based learning (PBL) into secondary mathematics teachers' training significantly improved their practical skills and pedagogical knowledge and also allowed them to reflect on their beliefs and approaches to teaching mathematics, promoting deeper and more meaningful learning. The study by [10] highlights how PBL can foster curiosity, proactivity, and reflective thinking in students while also enhancing their understanding of mathematical concepts and practical skills. In addition, the achievements and challenges faced by teachers in implementing this methodology are discussed, emphasizing the importance of collaboration and active student participation in the learning process.

Problem-Based Learning (PBL) has been applied innovatively in various academic disciplines. This is the case in the medical field, where students work in actual clinical cases to diagnose and propose treatments, helping them integrate theoretical knowledge into practical situations [11, 12]. In engineering programs, real-world problems are presented that require technical solutions [13]. In social sciences, PBL can involve the analysis of complex social issues, where students investigate, discuss, and propose solutions based on data and social theories[14].

A fundamental tool that allows teachers to assess the effectiveness and relevance of educational strategies is the use of evaluation instruments that measure both the implementation process and the results obtained. However, the use of meta-analyses and systematic reviews can reveal trends and practical strategies that generate consistent results in different contexts, resolve discrepancies between studies, and support evidence-based decision-making. However, improving teaching through the use of digital technology can be considerably limited by resource restrictions in certain institutions, which hinders the widespread adoption of these teaching strategies. [15]. The lack of technological infrastructure, such as computers or electronic devices necessary for students and teachers to use digital tools and limited Internet access, particularly in rural or marginalized areas, hinders the use of online platforms [16]. Furthermore, there is a lack of teacher training to integrate technology into their pedagogical practices [17]. Some strategies that can be implemented to mitigate these issues include the use of accessible technologies, such as open source software, refurbished devices, or applications optimized for low-connectivity environments. Likewise, prioritizing teacher training to use available digital resources [18] effectively. Addressing these barriers through an inclusive and sustainable approach can help ensure that the benefits of educational technologies reach all contexts, promoting a more equitable and quality education.

This paper aims to conduct a systematic review to better understand how learning difficulties in linear algebra are formally diagnosed and what teaching strategies are being implemented. The importance of this review becomes evident when considering that linear algebra is a fundamental subject in science and engineering courses. Using linear models to predict and control system behaviors contributes significantly to developing students' logical, heuristic, and algorithmic thinking skills.

Therefore, this review will analyze current knowledge on the diagnosis of student learning problems in linear algebra and the recent implementation of didactic strategies to improve teaching and learning in this field.

2 Method

Our search strategy used the PRISM protocol (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) as a reference and followed specific indicators [19]. We guided our research with four key questions.

1. What are the main factors influencing learning problems in linear algebra?
2. Which learning theories have been applied to formally diagnose these learning problems and design teaching strategies for linear algebra?
3. What are the developed thematic strategies for linear algebra, and do they share any common characteristics?
4. What were the sizes of the groups used to validate the formal diagnoses or as pilot groups for implementing teaching strategies?

To address our research questions and achieve the study's objective, we conducted a systematic literature review. This method is known to systematically integrate empirical results related to a specific research problem [20]. We developed our research methodology in four distinct stages, which we detail in the following paragraphs.

2.1 Stage 1: Setting Inclusion and Exclusion Criteria for Research Studies

In this first stage, we established specific criteria for including and excluding studies in our research. For inclusion, we focused on research articles, excluding other document types such as theses and book chapters. We considered articles published between 2013 and 2022, ensuring that the research was no more than 10 years old. In addition, we included studies written in Spanish, English, or Portuguese. The final inclusion criterion was that the articles must be related to teaching or learning linear algebra; we excluded articles on topics outside this specific educational area.

For exclusion, we omitted any articles that did not meet all our inclusion criteria. This also included articles that were duplicated in our study.

2.2 Stage 2: Developing the Search Strategy

In this stage, we executed our search strategy across various databases, obtaining 84 articles for analysis. Our search criteria varied according to the database to maximize results (see Figure 1). We selected databases that showed the highest number of relevant results for our topic. The databases and their respective search formulas were as follows:

- ERIC: Using the formula ("Education") AND ("Linear Algebra"), we obtained 9 articles.
- Scielo and DOAJ: We used ("Education") AND ("Linear Algebra") and ("Education") AND ("Linear Algebra"), obtaining 8 and 30 articles, respectively.
- Redalyc: With the formula ("Education") AND ("Linear Algebra"), we found 8 articles.
- Science Direct: We used (Teaching OR Learning) AND ("Linear Algebra"), leading to 9 articles.
- Dialnet: The formulas (Teaching OR Learning) AND ("Linear Algebra") and (Didactics) AND ("Linear Algebra") resulted in 7 articles.
- Scopus: Using the formula (Teaching OR Learning) AND ("Linear Algebra"), we found 13 articles.

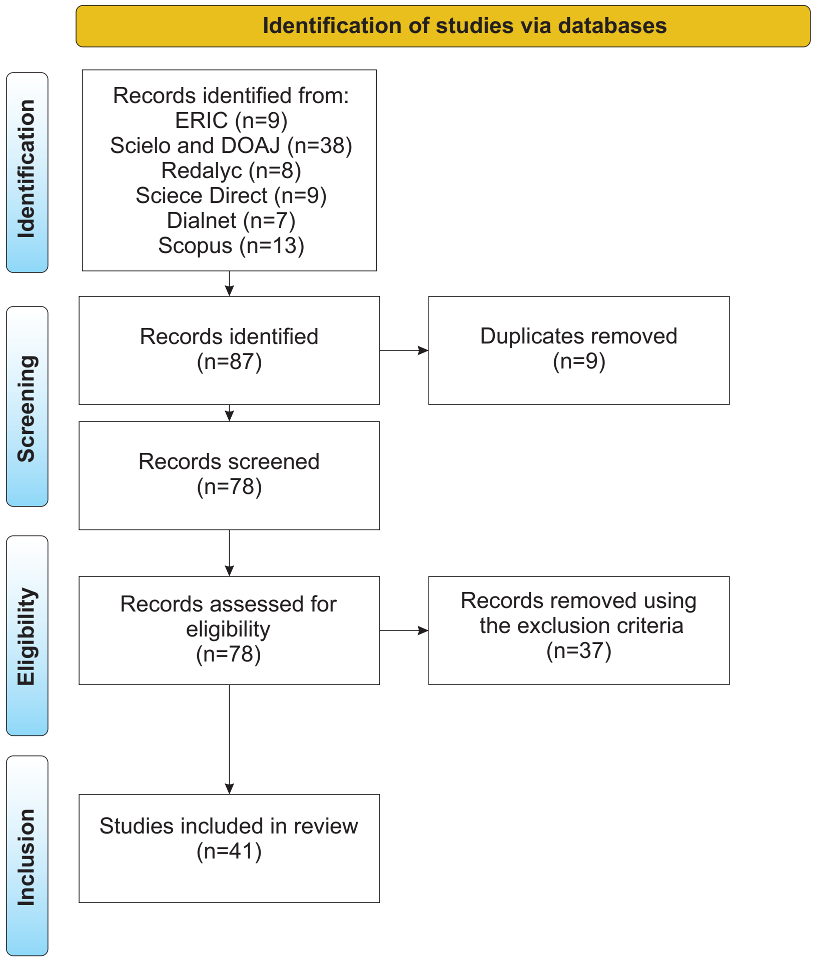

Figure 1: Overview of the Information Search and Data Collection Process This flowchart details the search terms used across various databases, the number of articles retrieved from each, and the filtering process leading to the final selection of articles included in the review. It also outlines the exclusion criteria applied and the total number of articles analyzed.

2.3 Stage 3: Information Purification

In this stage, we conducted an initial review of the 84 articles gathered from the databases and references mentioned above. The purpose of this review was to assess the relevance of each article to our research objectives. We rejected 34 articles during this process because they did not provide relevant data for our systematic review analysis or contribute to answering our research questions. Consequently, 27 articles were selected and included in our review.

2.4 Stage 4: Data Coding and Analysis

In this final stage, we analyzed the data based on specific categories. This structured approach helped us to thoroughly examine and understand the findings. The categories we focused on were:

1. Factors influencing learning problems in linear algebra.
2. Learning theories applied to diagnosing learning problems or implementing linear algebra teaching strategies.
3. Thematic contents within the subject of linear algebra were the focus of the investigation.
4. Strategies implemented in teaching linear algebra.
5. Sizes of the samples used for validation or implementation in pilot tests.

This categorization facilitated a comprehensive analysis of the collected data, aligning it closely with our research objectives.

3 Results

3.1 Factors Influencing Linear Algebra Learning Problems

The factors identified that influence learning problems in linear algebra are varied, as observed in the systematic review of the research. Despite this diversity, there is notable consistency in the findings. This is apparent when we see that several factors recur in multiple studies. In some cases, more than one factor is repeated between different investigations, as detailed in Table 1. This repetition highlights the commonalities in the challenges faced by linear algebra learners.

In our systematic review, we found that the most significant factor affecting linear algebra learning, as identified by various authors, is the level of abstraction and the formalism of the subject (see Figure 2). The high level of abstraction required by linear algebra itself poses a challenge for students, demanding a substantial degree of abstract thinking for proper understanding [47]. In terms of formalism, it comes from the way linear algebra is presented, studied, and learned in the literature, which is heavily based on the formalism of mathematical language [42].

Other key aspects impacting linear algebra learning difficulties include students' challenges in differentiating between a concept and its various representations [25] and the use of diverse languages when discussing vector spaces and linear transformations [31]. Additionally, the connection to the teacher's training emerges as a notable factor. If a teacher has a background in mathematics or a related field, the issue often lies in not having the foundational structures in place. Conversely, for engineering educators, the challenge is often linking the relevance and applicability of linear algebra concepts to their specific field [48].

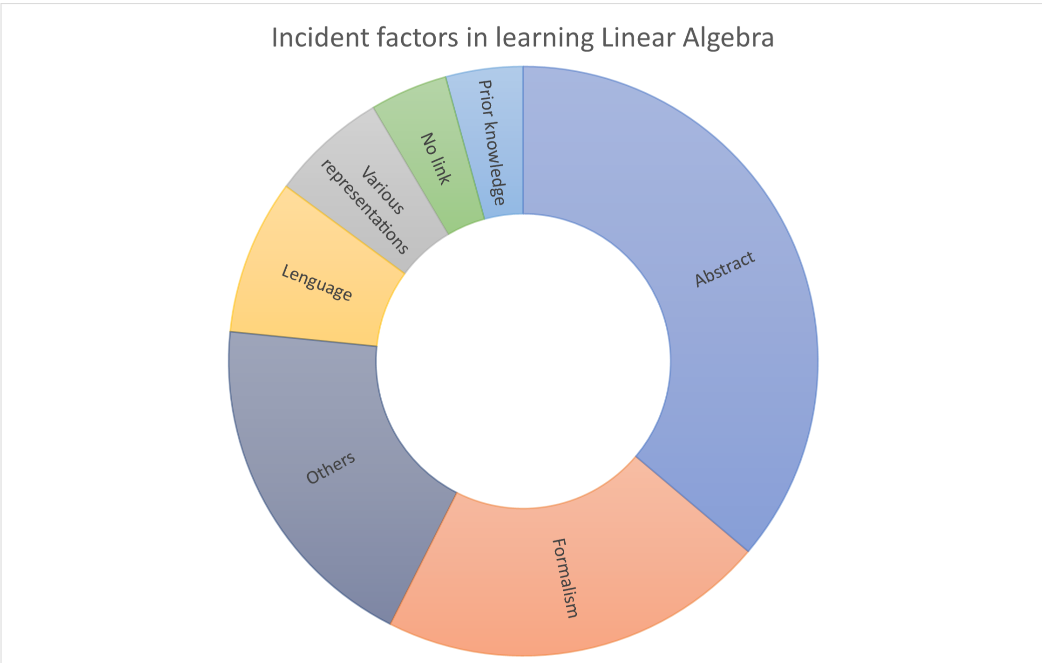

Figure 2: Prevalence of Factors Impacting Learning in Linear Algebra This bar chart illustrates the frequency of various factors that influence the learning of linear algebra, as identified in the reviewed research, including abstraction, formalism, language, and others.

3.2 Learning Theories Applied in Linear Algebra

This section highlights the learning theories applied to the diagnosis of learning difficulties and the implementation of didactic strategies in linear algebra. It also covers the tools used in the various research projects analyzed. In addition, we provide information about the countries where each study was conducted, as detailed in Table 2.

In the systematic review, the APOE theory emerges as the learning theory most frequently applied in the analyzed research works (see Figure 3). This theory has been used predominantly to diagnose learning problems in linear algebra more accurately and deeply. Using genetic decomposition, students develop mental schemes or structures that aid them in constructing knowledge about specific concepts [49].

Regarding the theory of semiotic representations, the reviewed studies have utilized it to support didactic strategies. These strategies involve varying representations of concepts, often enhanced by computational

Table 1: Influential Factors in Linear Algebra Learning Problems Identified in Scholarly Research. This table compiles pivotal studies on linear algebra, listing the year of publication, authors, article title, and the predominant factor influencing learning difficulties as identified in each piece of research.

Year of publication	Authors	Article	Predominant Factor
2013	Nishizawa et al. [21]	Increasing Reality and Educational Merits of a Virtual Game	Abstract
2013	Parraguez [22]	The role of the body in the construction of the concept of Vector Space	Abstract
2013	Rosso & Barros [23]	A taxonomy of errors in learning vector spaces	Abstract, Language, Various representations
2014	Birinci et al. [24]	University students' solution processes in systems of linear equation	Abstract, Prior knowledge, Axiomatic
2014	Ramírez-Sandoval et al. [25]	Coordination of semiotic representation records in the use of linear transformations in the plane	Various representations
2014	Salgado & Trigueros Gaisman [26]	A teaching experience of values, vectors and eigenspaces based on APOE theory	Abstract
2015	Trigueros Gaisman et al. [27]	Constructions and mental mechanisms for learning the matrix theorem associated with linear transformation	Abstract
2016	Berman & Shvartsman [28]	Definitions are important: the case of linear algebra	Formalism
2016	Marins & Pereira [29]	Advanced mathematical thinking manifested in tasks involving linear transformations	Formalism, Abstract
2017	Beltrán et al. [30]	Teaching Proposal for the Study of Eigenvectors and Eigenvalues.	Abstract, Formalism
2017	Costa & Rossignoli [31]	Teaching linear algebra in an engineering school: Methodological and didactic aspects	Abstract, Without connection with other subjects, Language
2018	Pierri [32]	From Practical to Theorical Thinking: The Impact of the Role-Play Activity.	Abstract, Formalism
2018	Afriza et al. [33]	The use of the wxMaxima linear algebra module on Gauss elimination lesson for mathematics education students	Concepts
2019	Álvarez-Macea & Costa[34]	Teaching Linear Algebra in engineering courses: an analysis of the process of mathematical modeling within the framework of the Anthropological theory of didactics	Epistemological component, didactic schemes, Language
2019	Aytekin & Kiymaz [35]	Teaching Linear Algebra Supported by GeoGebra Visualization Environment	Abstract, procedures memorization, lack of vinculation
2019	Gallo et al. [36]	Interpretation of linear transformations in the plane using GeoGebra	Formalism, Language, Various representations
2019	García-Hurtado et al. [37]	Linear algebra learning focused on plausible reasoning in engineering programs	Formalism
2019	Chérrez et al. [38]	Teaching-Learning of Matrices in the civil Engineering Course	Abstract, Formalism, Prior knowledge
2019	Novtiar et al. [39]	Development of innovative teaching and learning module in linear algebra course assisted by Maple	Complexity of interconnected concepts, the need for systematic thinking
2019	Stweart et al. [40]	Linear algebra teaching and learning: themes from recent research and evolving research priorities	Abstract concepts, theoretical aspects, reasoning and problem-solving
2019	Yang et al. [41]	Case design of linear algebra hybrid teaching model under problem-based learning	Complexity of abstract concepts, difficulty in visualizing vector spaces, challenges in understanding matrix operations
2020	Parraguez[42]	Construction of the meanings of vector space operations through linearly independent/dependent sets	Abstract, Formalism
2020	Pizarro [43]	A Didactic Sequence for Teaching Linear Transformation: Unification of Methods and Problems, Modeling and Explanation of Learning	Concept application conditions
2021	Cárcamo et al. [44]	Hypothetical learning trajectories: an example in a linear algebra course	Abstract
2021	Kariadinata[45]	Students Reflective Abstraction Ability on Linear Algebra Problem Solving and Relationship with Prerequisite Knowledge.	Abstract
2021	Silva et al.	Creation and uses of LineAlg application as a learning object in basic education	Formalism
2021	Wibawa et al. [46]	Learning Effectiveness Through Video Presentations and WhatsApp Group (WAG) in the Pandemic Time Covid-19	Abstract, Demonstrations, Large number of operations between variables

tools for a better graphic representation [50]. The anthropological theory of didacticism was applied to

Table 2: Influential Factors in Linear Algebra Learning Problems Identified in Scholarly Research. This table compiles pivotal studies on linear algebra, listing the year of publication, authors, article title, and the predominant factor influencing learning difficulties as identified in each piece of research.

Author and year	**Country**	**Applied learning theory**	**Tool used**
(Parraguez, 2013)	Chile	APOE	Semi-structured interview
(Rosso & Barros, 2013)	Argentina	Theory of didactic situations and constructivism	Problems situations
(Parraguez & Uzuriaga, 2014)	Chile	APOE	Questionnaire and interviews
(Ramírez-Sandoval et al., 2014)	México	Theory of semiotic representations	Interview with sequence of 5 activities
(Salgado & Trigueros Gaisman, 2014)	México	APOE	Questionary and semi-structured interview
(Trigueros Gaisman et al., 2015)	Chile	APOE	Questionary and semi-structured interview
(Murillo & Beltrán, 2016)	Spain	APOE	RGB color system
(González & Roa, 2017)	Colombia	APOE	Internalization of concrete actions
(Roa-Fuentes & Parraguez, 2017)	Chile and Colombia	APOE	Questionary
(Costa, 2018)	Argentina	Anthropological Theory of the Didactic	Study and research activity
(Karrer, 2018)	Brazil	Theory of semiotic represetations	Using GeoGebra
(Rodríguez et al., 2018)	Chile	APOE	Questionnaire and interviews
(Afriza et al., 2018)	Indonesia	ADDIE model	wxMaxima Software
(Rodríguez et al., 2018)	Chile	APOE	Questionnaire and interviews
(Álvarez-Macea & Costa, 2019)	Colombia	Anthropological Theory of the Didactic	Study and research activity
(Gallo et al., 2019)	Argentina	Theory of semiotic representations	Series of computer activities using GeoGebra software
(Novtiar et al. 2019)	Indonesia	Constructivism	Problem solving through theory and the use of Maple software
(Stweart et al. 2019)	Not specified	Cognitive resources, dynamic environments for visualization	web-based modules, interactive worksheets
(Yang et al. 2019)	China	Problem-based learning (PBL)	Traditional classroom teaching with online resources
(Parraguez, 2020)	Chile	APOE	Written questionnaire
(Fortuny & Fuentealba, 2021)	Spain	Realistic mathematics education	Guía escrita, archivos de audio y video, entrevistas con algunos estudiantes
(Betancur et al., 2022)	Colombia	APOE	Questionary and semi-structured interview

identify the learning difficulties of students in linear algebra and to support didactic strategies using modeling, incorporating technology such as mobile devices and software [51].

The didactic situation theory was used to categorize common errors in learning the topic of vector spaces [23]. Additionally, the column labeled ”others” in Figure 3 includes various theories such as the theory of didactic proposal situations and realistic mathematical education [44]. These theories have been instrumental in supporting didactic proposals for teaching linear algebra.

In the systematic review, we noted the tools used for conducting research. Among these are questionnaires and interviews, particularly in studies implementing the APOE theory. The GeoGebra software stands out, along with the use of study guides on virtual platforms and a variety of activities grounded in learning theories.

It is also worth noting the global reach of research in the field of linear algebra education. Chile emerges as a leader in research production in Latin America. However, countries outside the American continent, such as Spain, Turkey, and Indonesia, also contribute significantly. This emphasizes the universal relevance of the challenges in teaching and learning linear algebra, indicating that these difficulties are common in classrooms worldwide, regardless of location.

Regarding linear algebra teaching strategies, the review also examined the specific topics of the research that have been the focus of the research and the sample sizes used in these studies (see Table 3).

This review reveals a strong emphasis on the use of digital technology in teaching the topics discussed, with the specific tools and elements varying according to the research objectives (Figure 4). For example,

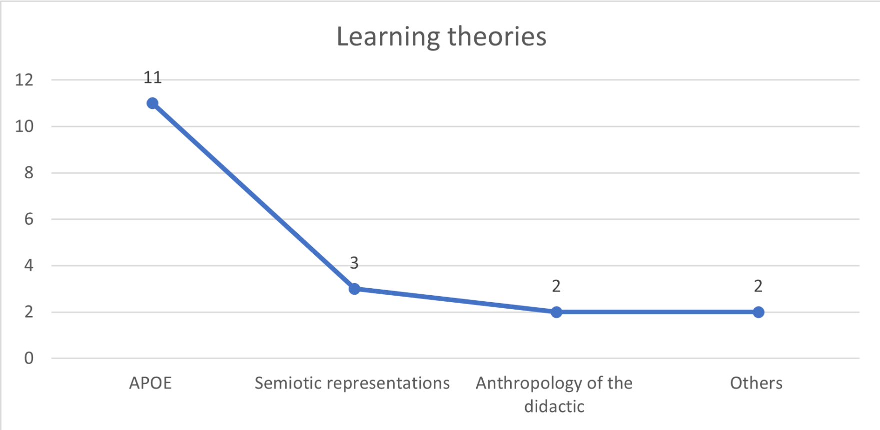

Figure 3: Distribution of Learning Theories in Reviewed Research. This figure illustrates the prevalence of different learning theories as applied in the research works reviewed. The APOE theory leads in application, followed by semiotic representations, the anthropological theory of the didactic, and other various theories.

there is a focus on the use of various mathematical software, knowledge management platforms [52], web-based learning tools [53], virtual games [21], and virtual evidence portfolios [54].

The systematic and thorough diagnosis of the mental structures that support the understanding of vector space concepts related to the design of the proposed activities was clearly observed in the study by [26]. However, a common thread across many studies is that topics of higher complexity and abstraction are most frequently addressed, both in diagnostic processes and in methodological proposals for teaching and learning.

In particular, studies targeting instruction within the domain of engineering, particularly mathematical modeling, are prominent [55]. This aligns with the practical application requirements characteristic of engineering curriculums.

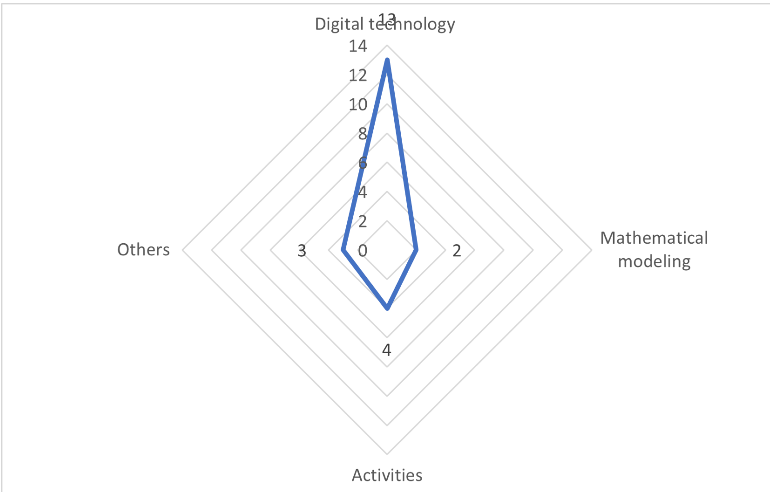

Figure 4: Frequency of Different Teaching Models or Strategies Used. This bar graph illustrates the frequency with which various teaching models or strategies are applied in linear algebra education, showcasing a predominant use of digital technology, followed by mathematical modeling, diverse learning activities, and other strategies.

The systematic review of research works revealed that most teaching strategies and diagnostic efforts in linear algebra are focused on more abstract concepts. Vector spaces [56], linear transformations [49], and matrices are the topics most frequently addressed. Less commonly, but still noteworthy, are studies on systems of linear equations [24] and eigenvalues and eigenvectors [57]. These findings align with the goal of the research: to develop tools that mitigate the factors impacting the teaching and learning of complex linear algebra topics [28].

Table 3: Overview of Teaching Models or Strategies, Topics, and Sample Sizes in Linear Algebra Research. This table details the teaching models or strategies applied to linear algebra topics, specifying the topics addressed and the sample sizes involved in each study.

Author and year	**Model or strategy**	**Topics**	**Sample size**
(Nishizawa et al., 2013)	Digital technology	Vectors in 3D	40 students
(Yildiz Ulus, 2013)	Digital technology	Eigenvectors and eigenvalues	Not implementation
(Salgado & Trigueros Gaisman, 2014)	APOE-based activities	Eigenvectors and eigenvalues	34 students on average per semester
(Petrov et al., 2015)	Digital technology	Matrices and determinants, Vector spaces, Eigenvectors and eigenvalues	37 students
(Gabriel Vergara et al., 2016)	Digital technology	Systems of linear equations, Matrices, Eigenvectors and eigenvalues	35 teachers and 5 students
(Murillo & Beltrán, 2016)	Digital technology	Vector spaces	Not implementation
(Torres et al., 2016)	Digital technology	Systems of linear equations, Vector spaces, Matrices, Linear transformations, Eigenvectors and eigenvalues	Not implementation
(Costa &Rossignoli, 2017)	Digital technology	Not specified	Voluntaries 295 students
(Meneu et al., 2017)	Activities	Eigenvectors and eigenvalues	Not implementation
(Costa, 2018)	Digital technology	Linear algebra with physics	50 students
(Karrer, 2018)	Digital technology	Linear transformations	2 students
(Karrer, 2018)	Digital technology	Linear transformations	2 students
(Kartika et al., 2018)	Digital technology	Vectors 3D	69 students
(Pierri, 2018)	Digital technology	Systems of linear equations, Matrices, Vector spaces	70 students
(Afriza et al., 2018)	Digital technology	Solutions of homogeneous linear equation systems, and the concept of Gauss Jordan elimination method	18 students
(Aytekin & Kiymaz, 2019)	Digital technology	Vector spaces	4 students
(Gallo et al., 2019)	Digital technology	Linear transformations	Not implementation
(García-Hurtado et al., 2019)	Mathematical modeling	System of linear equations, Matrices and determinants, Vectors, Vector spaces	36 students
(Villalobos & Ríos, 2019)	Digital technology	Vector operations	40 students
(Xavier et al., 2019)	Activities	Matrices	Not implementation
(Novtiar et al. 2019)	Digital technology, active learning	Determinants and Inverse Matrices, Linear Equation Systems, Vector Space, Inner Product Space, Linear Transformation, and Eigenvalue	Not specified
(Stweart et al. 2019)	Digital technology	Leveraging geometric intuitions	Group of 144 teacher trainees
(Yang et al. 2019)	Problem-driven teaching methods, pre-class preparation with online resources, group discussions	Vector spaces and linear transformations, Matrices and matrix operations, linear equations and systems of linear equations, determinants and their properties, eigenvalues and eigenvectors, Applications of linear algebra in real-world problems	Not specified
(Nissa et al., 2020)	Problem-based learning	Systems of linear equations, Matrices	21 students and 21 control group
(Pizarro, 2020)	Didactic engineering and Mathematical modeling	Linear transformations	17 students
(Fortuny & Fuentealba, 2021)	Hypothetical learning trajectories	Vector spaces	7 students
(Silva et al., 2021)	Digital technology	Matrices, systems of linear equations	Not implementation
(Wibawa et al., 2021)	Digital technology	Vector spaces	14 students

4 Discussion and Conclusions from the Systematic Review

The systematic review has led to several important conclusions regarding the factors that hinder students' learning of linear algebra. High levels of abstraction [58], unfamiliar formalism [29], language barriers [34], multiple representations of mathematical objects [36], lack of prior knowledge [46], and weak connections in learning [29] are significant challenges. Additionally, the complexity of new definitions, the number of operations between variables, and the subject's epistemological and axiomatic characteristics are noted as less frequent but still impactful factors. In terms of learning theories, the review accentuates the APOE theory as the predominant framework for in-depth research on learning difficulties in linear algebra. The popularity of the theory suggests that it effectively uncovers and addresses the mental structures of students during the construction of knowledge, as highlighted by Rodriguez et al. [5]. Despite this, the main application of APOE theory is in diagnosis, with other theories more commonly used to explore the results of various teaching and learning strategies, except in the work of Salgado and Trigueros [26]. This review reveals a gap: the direct link between systematic diagnosis and strategy application is often absent. This could be due to the urgent need for educational institutions to produce quick results, relying on the experience and conceptual understanding of the authors to design their approaches. The role of digital technology is consistently significant in the research on teaching and learning strategies. Mathematical software applications [50], [59], [60], web-based learning tools especially relevant during the COVID-19 pandemic for remote education, and virtual games [27] are some examples that reflect the growing and irreversible trend of digital integration in education. The main focus of research in terms of content includes vector spaces [45] and linear transformations [61], likely due to their complex and abstract nature requiring deep understanding. Regarding sample sizes for statistical analysis in the reviewed studies, they ranged from 2 to 295 participants, with variations in application time and student nationalities. This indicates a need for further research with larger populations, leveraging digital technology for more extensive validation and evaluation. The reviewed research, regardless of its focus, often bases some methodological aspects on the authors' experiences, their conceptual understanding, and sometimes the influence of a research community. The effectiveness of proposed solutions is most significantly validated by the experiences of those who implement them. Therefore, future research should aim to enhance the authors' experiences and perspectives by developing methodologies that better connect with research communities and employ digital technology. This approach could allow a broader student population to participate and benefit from the methodologies proposed in this review. Regarding the learning of Algebra from its basic concepts to complex topics, a gradual approach can be designed to structure progressive teaching by starting with fundamental principles and advancing to abstract concepts. As in other disciplines, including practical examples and real applications helps students understand how complex theories relate to real situations, increasing their motivation and comprehension. The integration of active learning strategies such as gamification, collaborative problem-solving, simulations, and practical experiments allows for reinforcing basic concepts while introducing abstract topics. This improves knowledge retention and encourages active student participation. Encouraging continuous feedback allows students to understand the topics better and provides an opportunity to adjust teaching strategies to ensure that student needs are met in a timely manner, preventing them from falling behind. This maximizes the practical utility of the proposed strategies and improves the learning experience.

Acknowledgment

We want to express our sincere gratitude to CONAHCYT for providing the scholarship that supported the graduate studies that enabled this investigation. Their generous assistance was invaluable to the completion of this project.

References

[1] E. Possani, M. Trigueros, J. Preciado, and M. D. Lozano, "Use of models in the teaching of linear algebra," *Linear Algebra and its Applications*, vol. 432, no. 8, pp. 2125–2140, 2010.

[2] M. Trigueros and M. Wawro, "Linear algebra teaching and learning," *Encyclopedia of mathematics education*, pp. 474–478, 2020.

[3] E. Bagno, T. Dana-Picard, and S. Reches, "Chatgpt in teaching linear algebra: Strides forward, steps to go," *Open Education Studies*, vol. 6, no. 1, p. 20240031, 2024.

[4] A. Oktaç, "Conceptions about system of linear equations and solution," *Challenges and strategies in teaching linear algebra*, pp. 71–101, 2018.

[5] M. Rodríguez Jara, M. Parraguez González, and M. Trigueros Gaisman, "Construcción cognitiva del espacio vectorial r 2," *Revista latinoamericana de investigación en matemática educativa*, vol. 21, no. 1, pp. 57–86, 2018.

[6] B. Wikieł and J. Mytnik, "Colossus from algebron–gamification as a tool to engage students in learning mathematics," *Scientific Journal of Gdynia Maritime University*, no. 131, pp. 59–72, 2024.

[7] A. G. G. Peralta and M. S. Aguilar, "A linear algebra quiz game as a supplementary learning tool in tertiary-level." *Pedagogical Research*, vol. 9, no. 1, 2024.

[8] I. Vale and A. Barbosa, "Active learning strategies for an effective mathematics teaching and learning," 2023.

[9] J. Bosica, J. S. Pyper, and S. MacGregor, "Incorporating problem-based learning in a secondary school mathematics preservice teacher education course," *Teaching and Teacher Education*, vol. 102, p. 103335, 2021.

[10] S. Rézio, M. P. Andrade, and M. F. Teodoro, "Problem-based learning and applied mathematics," *Mathematics*, vol. 10, no. 16, p. 2862, 2022.

[11] J. C. Trullàs, C. Blay, E. Sarri, and R. Pujol, "Effectiveness of problem-based learning methodology in undergraduate medical education: a scoping review," *BMC medical education*, vol. 22, no. 1, p. 104, 2022.

[12] K. T. Challa, A. Sayed, and Y. Acharya, "Modern techniques of teaching and learning in medical education: a descriptive literature review," *MedEdPublish*, vol. 10, 2021.

[13] V. Sukackė, A. O. P. d. C. Guerra, D. Ellinger, V. Carlos, S. Petronienė, L. Gaižiūnienė, S. Blanch, A. Marbà-Tallada, and A. Brose, "Towards active evidence-based learning in engineering education: A systematic literature review of pbl, pjbl, and cbl," *Sustainability*, vol. 14, no. 21, p. 13955, 2022.

[14] E. Groenewald, O. K. Kilag, R. Unabia, M. Manubag, M. Zamora, and D. Repuela, "The dynamics of problem-based learning: A study on its impact on social science learning outcomes and student interest," *Excellencia: International Multi-disciplinary Journal of Education (2994-9521)*, vol. 1, no. 6, pp. 303–313, 2023.

[15] A. C. Umaña-Mata, "Educación superior en tiempos de covid-19: oportunidades y retos de la educación a distancia," *Revista Innovaciones Educativas*, vol. 22, pp. 36–49, 2020.

[16] J. Rodríguez, D. Marín, S. López, and M. Castro Rodríguez, "Tecnología y escuela rural: avances y brechas," *EICE. Revista Iberoamericana Sobre Calidad, Eficacia Y Cambio En Educación*, vol. 21, no. 3, pp. 139–157, 2023.

[17] C. A. C. Olivera and R. d. l. C. A. Rosell, "Desafíos de la capacitación docente orientada a las tecnologías de la información y las comunicaciones (tics). revisión sistemática," *Ciencia Latina Revista Científica Multidisciplinar*, vol. 7, no. 3, pp. 2427–2456, 2023.

[18] L. Castro-Palomino and E. A. Coras, "Herramientas digitales en el desempeño de los docentes: revisión sistemática," *Horizontes. Revista de Investigación en Ciencias de la Educación*, vol. 8, no. 32, pp. 288–299, 2024.

[19] J. J. Yepes-Nuñez, G. Urrutia, M. Romero-Garcia, and S. Alonso-Fernandez, "The prisma 2020 statement: an updated guideline for reporting systematic reviews declaración prisma 2020: una guía actualizada para la publicación de revisiones sistemáticas," *Revista espanola de cardiologia*, vol. 74, no. 9, pp. 790–799, 2021.

[20] J. Sánchez Meca, "Cómo realizar una revisión sistemática y un meta-análisis," *Aula abierta*, vol. 38, no. 2, pp. 53–64, 2010.

[21] H. Nishizawa, K. Shimada, W. Ohno, and T. Yoshioka, "Increasing reality and educational merits of a virtual game," *Procedia Computer Science*, vol. 25, pp. 32–40, 2013.

[22] M. Parraguez González, "The role of the field in the construction of the concept of vector space," *Educación Matemática*, vol. 25, pp. 133–154, 01 2013.

[23] A. Rosso, J. Barros *et al.*, "Una taxonomía de errores en el aprendizaje de espacios vectoriales," *Revista iberoamericana de educación*, 2013.

[24] D. K. Birinci, A. Delice, and E. Aydın, "University students solution processes in systems of linear equation," *Procedia-Social and Behavioral Sciences*, vol. 152, pp. 563–568, 2014.

[25] O. Ramírez-Sandoval, C. F. ROMERO-FÉLIX, and A. OKTAÇ, "Coordinación de registros de representación semiótica," in *Annales de Didactique et de Sciences Cognitives*, 2014, p. 225.

[26] H. Salgado and M. Trigueros, "Una experiencia de enseñanza de los valores, vectores y espacios propios basada en la teoría apoe," *Educación matemática*, vol. 26, no. 3, pp. 75–107, 2014.

[27] M. Trigueros Gaisman, I. Maturana Peña, M. Parraguez González, and M. Rodríguez Jara, "Construcciones y mecanismos mentales para el aprendizaje del teorema matriz asociada a una transformación lineal," *Educación matemática*, vol. 27, no. 2, pp. 95–124, 2015.

[28] A. Berman and L. Shvartsman, "Definitions are important: The case of linear algebra." *European Journal of Science and Mathematics Education*, vol. 4, no. 1, pp. 26–32, 2016.

[29] A. S. Marins and A. M. P. d. D. Savioli, "Pensamento matemático avançado manifestado em tarefas envolvendo transformações lineares," *Ciência & Educação (Bauru)*, vol. 22, pp. 489–504, 2016.

[30] M. J. M. A. Beltran Meneu and E. Marina Jorda Mora, "A teaching proposal for the study of eigenvectors and eigenvalues," *Journal of Technology and Science Education*, 2017.

[31] V. A. Costa and R. Rossignoli, "Enseñanza del algebra lineal en una facultad de ingeniería: Aspectos metodológicos y didácticos," *Revista Educación en Ingeniería*, vol. 12, 2017.

[32] A. Pierri, "From practical to theoretical thinking: The impact of the role-play activity," *Journal of e-Learning and Knowledge Society*, vol. 14, no. 3, 2018.

[33] D. Afriza, C. Zubainur, and A. Veloo, "The use of the wxmaxima linear algebra module on gauss elimination lesson for mathematics education students," in *Journal of Physics: Conference Series*, vol. 1088, no. 1. IOP Publishing, 2018, p. 012088.

[34] F. Álvarez-Macea and V. A. Costa, "Enseñanza del algebra lineal en carreras de ingeniería: un análisis del proceso de la modelización matemática en el marco de la teoría antropológica de lo didáctico." *Eco Matemático*, vol. 10, no. 2, pp. 65–78, 2019.

[35] C. Aytekin and Y. Kiymaz, "Teaching linear algebra supported by geogebra visualization environment." *Acta Didactica Napocensia*, vol. 12, no. 2, pp. 75–96, 2019.

[36] H. G. Gallo, C. A. Verón, and C. G. Herrera, "Interpretación de transformaciones lineales en el plano utilizando geogebra," *Revista Iberoamericana de Tecnología en Educación y Educación en Tecnología*, no. 24, pp. 32–37, 2019.

[37] O. García-Hurtado, M. M. García-Pupo, R. Poveda-Chaves *et al.*, "Linear algebra learning focused on plausible reasoning in engineering programs," *Visión electrónica*, vol. 13, no. 2, pp. 322–330, 2019.

[38] R. X. Cherrez Ibarra and M. Escalona Reyes, "The teaching process of matrixes in the civil engineering major from universidad laica eloy alfaro de manabi," vol. 5, pp. 51–60, 10 2019.

[39] C. Novtiar and A. Fitrianna, "Development of innovative teaching and learning module in linear algebra course assisted by maple," in *Journal of Physics: Conference Series*, vol. 1315, no. 1. IOP Publishing, 2019, p. 012050.

[40] S. Stewart, C. Andrews-Larson, and M. Zandieh, "Linear algebra teaching and learning: themes from recent research and evolving research priorities," *ZDM*, vol. 51, pp. 1017–1030, 2019.

[41] Y. Yang, Q. Zhang, B. Liu, L. Wang, and X. Kong, "Case design of linear algebra hybrid teaching model under problem-based learning," *International Journal of Information and Education Technology*, vol. 9, no. 9, pp. 618–622, 2019.

[42] M. Parraguez, "Construcción de significados de las operaciones del espacio vectorial a través de conjuntos linealmente independientes/dependientes," *RECHIEM. Revista Chilena de Educación Matemática*, vol. 12, no. 2, pp. 60–70, 2020.

[43] S. Pizarro, "A didactic sequence for teaching linear transformation: Unification of methods and problems, modeling and explanation of learning," *Revista Latinoamericana de Investigación en Matemática Educativa*, vol. 23, pp. 271–310, 2020.

[44] A. Cárcamo, J. M. Fortuny, C. E. Fuentealba Aguilera *et al.*, "Las trayectorias hipotéticas de aprendizaje: un ejemplo en un curso de álgebra lineal," *Enseñanza de las ciencias: revista de investigación y experiencias didácticas*, 2021.

[45] R. Kariadinata *et al.*, "Students'reflective abstraction ability on linear algebra problem solving and relationship with prerequisite knowledge," *Infinity Journal*, vol. 10, no. 1, pp. 1–16, 2021.

[46] K. A. Wibawa and I. Payadnya, "Learning effectiveness through video presentations and whatsapp group (wag) in the pandemic time covid-19," *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, vol. 10, no. 2, pp. 710–720, 2021.

[47] J. T. Torres, M. I. García-Planas, and S. Domínguez-García, "The use of e-portfolio in a linear algebra course." *Journal of Technology and Science Education*, vol. 6, no. 1, pp. 52–61, 2016.

[48] S. Pascual Pizarro, "Una secuencia didáctica para la enseñanza de la transformación lineal: unificación de métodos y problemas, modelización y explicitación del aprendizaje," *Revista latinoamericana de investigación en matemática educativa*, vol. 23, no. 3, pp. 271–310, 2020.

[49] S. Roa-Fuentes and M. Parraguez, "Estructuras mentales que modelan el aprendizaje de un teorema del álgebra lineal: Un estudio de casos en el contexto universitario," *Formación universitaria*, vol. 10, no. 4, pp. 15–32, 2017.

[50] M. Karrer and R. C. M. Santos, "Núcleo, imagem e composição de transformações lineares: uma abordagem gráfica desenvolvida em ambiente computacional," *Research, Society and Development*, vol. 7, no. 10, pp. e9 710 405–e9 710 405, 2018.

[51] V. A. Costa, "Uso de dispositivos móviles y de software matemático en la enseñanza por investigación," *Revista Electrónica de Enseñanza de las Ciencias*, vol. 17, 2018.

[52] P. Petrov, I. Gyudzhenov, and D. Tuparova, "Adapting interactive methods in the teaching of linear algebra–results from pilot studies," *Procedia-Social and Behavioral Sciences*, vol. 191, pp. 142–146, 2015.

[53] H. Kartika, "Teaching and learning mathematics through web-based resource: An interactive approach," *MaPan: Jurnal Matematika dan Pembelajaran*, vol. 6, no. 1, pp. 1–10, 2018.

[54] T. R. da Silva, W. L. F. dos Santos, E. R. B. da Silva, and A. W. A. de Arruda, "Criação e usos do aplicativo linealg como objeto de aprendizagem na educação básica," *Diversitas Journal*, vol. 6, no. 1, pp. 1415–1427, 2021.

[55] O. G. Hurtado, M. M. G. Pupo, and R. M. P. Chaves, "Linear algebra learning focused on plausible reasoning in engineering programs," *Visión electrónica*, vol. 13, no. 2, p. 14, 2019.

[56] M. P. González and V. L. U. López, "Construcción y uso del concepto combinación lineal de vectores," *Scientia et technica*, vol. 19, no. 3, pp. 328–334, 2014.

[57] M. J. B. Meneu, M. M. Arcila, and E. J. Mora, "A teaching proposal for the study of eigenvectors and eigenvalues," *JOTSE*, vol. 7, no. 1, pp. 100–113, 2017.

[58] M. P. González, "El rol del cuerpo en la construcción del concepto espacio vectorial," *Educación Matemática*, vol. 25, no. 1, pp. 133–154, 2013.

[59] M. J. Beltrán Meneu and M. Murillo Arcila, "Coloreando el álgebra lineal," *Modelling In Science Education and Learning*, vol. 9, no. 1, pp. 25–34, 2016.

[60] R. C. Ibarra and M. E. Reyes, "El proceso de ensenanza-aprendizaje de las matrices en la carrera de ingenieria civil de la universidad laica eloy alfaro de manabi," *Mikarimin. Revista Cientifica Multidisciplinaria*, vol. 5, no. 2, pp. 51–60, 2019.

[61] D. E. González Rojas and S. Roa Fuentes, "Un esquema de transformación lineal: construcción de objetos abstractos a partir de la interiorización de acciones concretas," *Enseñanza de las Ciencias*, vol. 35, no. 2, pp. 0089–107, 2017.